Static charged fluid spheres in general relativity

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Interior perfect fluid solutions for the Reissner-Nordström metric are studied on the basis of a new classification scheme. It specifies which two of the characteristics of the fluid are given functions and accordingly picks up one of the three main field equations, the other two being universal. General formulae are found for charged de Sitter solutions, the case of constant energy component of the energy-momentum tensor, the case of known pressure (including charged dust) and the case of linear equation of state. Explicit new global solutions, mainly in elementary functions, are given as illustrations. Known solutions are briefly reviewed and corrected.

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I. INTRODUCTION

The unique exterior metric for a spherically symmetric charged distribution of matter is the Reissner-Nordström solution. Interior regular charged perfect fluid solutions are far from unique and have been studied by different authors. The case of vanishing pressure (charged dust (CD)) has received considerable attention. The general solution, in which the fluid density equals the norm of the invariant charge density, was presented in curvature coordinates by Bonnor [1]. The proof that this relation characterizes regular CD in equilibrium, i.e. in the general static case, was given later [2,3]. In the spherically symmetric case another proof was proposed in Ref. [4]. Concrete CD solutions were studied in these coordinates [4,5]. The generalization of the incompressible Schwarzschild sphere to the charged case with constant T_0^0 was also undertaken in a CD environment [6]. Charged dust, however, has been investigated more frequently in isotropic coordinates, since these encompass the entire static case and allow to search for interior solutions to the more general Majumdar-Papapetrou electrovacuum fields [7,8]. In both coordinate systems there is a simple functional relation between g_{00} and the electrostatic potential. In isotropic coordinates there is one non-linear main equation [7,9] which has been given several spherical [10–13] and spheroidal [11,12,14] solutions. One of them coincides with the general static conformally flat CD solution [15]. These CD clouds may be realized in practice by a slight ionization of neutral hydrogen, although the necessary equilibrium is rather delicate. They have a number of interesting properties: their mass and radius may be arbitrary, very large redshifts are attainable, their exteriors can be made arbitrarily near to the exterior of an extreme charged black hole. In the spheroidal case the average density can be arbitrarily large, while for any given mass the surface area can be arbitrarily small. When the junction radius r_0 shrinks to zero, many of their characteristics remain finite and non-trivial. One can even entertain the idea for a point-like classical model of electron were it not for the unrealistic ratio of charge e to mass m [1].

Recently, new static CD solutions were found, in particular with density which is constant or is concentrated on thin shells [16,17]. In the spherically symmetric case a relation has been established with solutions of the Sine-Gordon and the $\lambda\phi^4$ equations [18].

The necessary condition for a quadratic Weyl-type relation has been derived also for perfect fluids with non-vanishing pressure [19,20]. However, in this case many other dependencies between the electrostatic and the gravitational potential are possible, even when combined with constant T_0^0 [21].

The original Schwarzschild idea of constant density has been also tested in the charged case for a perfect fluid [22–24] or for imperfect fluid with two different pressures [1]. An electromagnetic mass model with vanishing density has been proposed in Ref. [6]. Unfortunately, the fluid has negative pressure (tension). Although the junction conditions do not require the vanishing of the density at the boundary, this is true for gaseous spheres. A model with such type of density was proposed both in the uncharged and the charged case [25,26].

Another idea about the electromagnetic origin of the electron mass maintains that, due to vacuum polarization, its interior has the equation of state $\rho + p = 0$, where ρ is the density and p is the pressure. This leads to tension, easier junction conditions and realistic e and m [27–29]. It can be combined with a Weyl-type character of the field [30]. The experimental evidence that the electron's diameter is not larger than 10^{-16} cm, however, requires that the classical models should contain regions of negative density [31,32]. Probably an interior solution of the Kerr-Newman metric is more adequate in this respect.

The presence of five unknown functions and just three essential field equations allows one to specify the metric and solve for the fluid characteristics [33]. This is impossible in the uncharged case. Another approach is to electrify some

of the numerous uncharged solutions. This has been done for one of the Kuchowicz solutions [34] in Ref. [35]. Two other papers [36,37] build upon the Wyman-Adler solution [38,39]. Thus a charged solution is obtained, which has approximately linear equation of state when m/r_0 is small. In Ref. [40] a generalization of the Klein-Tolman (KT) solution [41,42] was performed, but the resulting fluid does not possess a linear equation of state. Recently, static uncharged stars with spatial geometry depending on a parameter [43,44] have been generalized to the charged case [45].

The purpose of the present paper is to present a new and simple classification scheme for charged static spherically symmetric perfect fluid solutions. The calculations in each case are pushed as further as possible and general formulae are given in many instances. The known solutions are reviewed and compartmentalized according to this scheme in order to illustrate general ideas, without being exhaustive. New solutions are added where appropriate. The intention has been to stick to the simplest cases and remain in the realm of elementary if not algebraic functions. The junction conditions and other requirements for physically realistic models are discussed. The emphasis is, however, on the general picture, which appears unexpectedly rich and simpler than in the uncharged case.

The metric of a static spherically symmetric spacetime in curvature coordinates reads

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\Omega^2, \tag{1}$$

where $d\Omega^2$ is the metric on the two-sphere and ν, λ depend on r. The fluid and its gravitation are described by five functions depending on the radius: λ, ν, ρ, p and the charge function q which measures the charge within radius r. There are only three essential field equations, hence, two of the above characteristics must be given. We shall classify solutions according to this feature. For example, (ν, λ) is the case of given metric and the other three fluid characteristics are found from the equations. This does not mean that solutions are distributed among groups which do not overlap. Thus, (ρ, q) is a completely general case - any solution, after ρ and q are known, may be put into this class. The essence is that ρ and q^2 are given functions and there is control over them; they can be chosen regular, positive and comparatively simple. Then the other three functions are usually more complex and are not always physically realistic.

In Sec.II the Einstein-Maxwell equations are organized into three main and two auxiliary ones. Two of the main equations are universal for all cases, the third one varies from case to case. Cases with given ν have linear first-order differential equations for $e^{-\lambda}$. Cases with given λ have linear second-order equations for $e^{\nu/2}$ or non-linear first order equations for $\rho + p$. These results hold also for fluids with a linear equation of state. The difficulties in the cases (ρ, p) and (ρ, q) are discussed, which prevent their analytical treatment. The junction conditions are given in general form and a reasonable set of physical requirements is included.

In Sec.III the case (λ, ν) is briefly discussed. More attention is paid to the condition $T_0^0 = const$ which is a case of given λ. The simplest algorithm for generating solutions is presented. It is based on a master function and involves simple operations. The same function is relevant also when ν is given.

In Sec.IV the case $\rho + p = 0$ is solved completely. It represents a charged generalization of de Sitter spacetime. One new solution is worked out as an example. This case is generalized to include solutions with positive pressure and an explicit example is given.

Sec.V deals with the case of given pressure which includes as a subcase charged dust solutions (they have vanishing pressure). Efforts are concentrated on the case (ν, p) which is solved in general, λ given as an explicit functional of the pressure. Conditions for regular density are elucidated. The well-known case of regular CD is used as an example to illustrate how the classification scheme works. Known solutions are reviewed both in curvature and isotropic coordinates. A new solution is given to demonstrate the features of a simple ansatz for ν, used with variations in the next sections, too. The most general CD solution is derived. It is not regular at the centre but can form a 'halo' around a typical solution with given pressure. An explicit example of such triple-layered model with realistic properties is applied.

The case (ν, q) is discussed in Sec.VI. It is the most direct generalization of the uncharged case. An integral representation for $e^{-\lambda}$ is given. A series of ansatze for ν is introduced in general, the first two cases are worked out explicitly. A new solution is presented and some known results are corrected.

In Sec.VII the important case (ν, n) of a fluid with linear equation of state, involving the parameters n and p_0, and given ν is investigated. Unlike the situation for uncharged fluid, the third main equation is linear and of first order. It allows a generic integral representation for $e^{-\lambda}$ which comprises also the cases (ν, q) and (ν, ρ). The structure of the density and its regularity are discussed which imply certain restrictions on ν. The solutions are in some way electrified versions of the Einstein static universe (ESU). A basic simplification technique leads to an infinite series of models with realistic physical properties. Two of them are studied in detail, including a model related to incoherent radiation. This kind of fluid appears also as a special case of the integral representation. Another special case is also soluble in elementary functions but has negative pressure and is closer to the cases studied in Sec.IV. Finally, a third special case, which is a twisted generalization of an obscure integrable uncharged case, is shown to be ill-defined in the presence of charge.

Sec.VIII is dedicated to the case (ν, ρ), closely related to (ν, n). A solution in terms of hypergeometric functions is found when the density is constant. An algebraic ansatz for the density, depending on two parameters, leads to realistic solution except for q^2 which turns out to be negative. The process of fine-tuning, however, may give truly realistic solutions in the future.

In Sec.IX the three cases (ρ, q), (λ, ρ) and (λ, q), which are closely related to each other, are discussed. It represents a critical review of Refs. [6,22,23,25,26]. A unified scheme is followed, which fully explores the connections between the different subcases. Several new solutions are obtained which demonstrate how hard is to escape the curse of negative pressure, which plagues purely electromagnetic mass models. The solution of Wilson [23] is formulated as a hypergeometric function.

In Sec.X the theme about a linear equation of state is raised once again, this time in the frame of case (λ, n). First, a natural generalization of the KT solution is obtained for constant λ, which is quite different from the solution in Ref. [40]. Then the ansatz for λ, which ensures that T_0^0 is constant, is studied in detail. The general solution of the second order linear equation is expressed once more through the hypergeometric function and degenerate cases in elementary functions are sought for. Three different classes are studied. One reduces to ESU. The second has nice orthogonal polynomial solutions for $e^{\nu/2}$ but with negative pressure, much like de Sitter solution which is contained as a subcase. This unpleasant feature is also shared by a trigonometric solution for the analog of incoherent radiation. The study of the non-linear first order equation produces even more bizarre results.

Sec. XI contains some discussion and conclusions. In all sections old solutions, known to the author, are reviewed and classified. New solutions have been checked by computer for realistic properties.

II. MAIN EQUATIONS AND CLASSIFICATION

The Einstein-Maxwell equations are written as

$$\kappa T_0^0 \equiv \kappa\rho + \frac{q^2}{r^4} = \frac{\lambda'}{r}e^{-\lambda} + \frac{1}{r^2}\left(1 - e^{-\lambda}\right), \tag{2}$$

$$\kappa p - \frac{q^2}{r^4} = \frac{\nu'}{r}e^{-\lambda} - \frac{1}{r^2}\left(1 - e^{-\lambda}\right), \tag{3}$$

$$\kappa p + \frac{q^2}{r^4} = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right), \tag{4}$$

where the prime means a derivative with respect to r and $\kappa = 8\pi G/c^4$. We shall use units where $G = c = 1$. The charge function is obtained by integrating the charge density σ. We shall use, however, q as a primary object and then

$$\kappa\sigma = \frac{2q'}{r^2}e^{-\lambda/2}. \tag{5}$$

Spherical symmetry allows only a radial electric field with potential ϕ given by

$$F_{01} = \phi' = -\frac{q}{r^2}e^{\frac{\nu+\lambda}{2}}. \tag{6}$$

Eq.(2) may be integrated by the introduction of the mass function

$$M(r) = \frac{\kappa}{2}\int_0^r r^2 T_0^0 dr = \frac{\kappa}{2}\int_0^r \left(\rho + \frac{q^2}{\kappa r^4}\right) r^2 dr, \tag{7}$$

and gives

$$z \equiv e^{-\lambda} = 1 - \frac{2M}{r}. \tag{8}$$

This can be rewritten as

$$\kappa\rho + \frac{q^2}{r^4} = \frac{2M'}{r^2} = \frac{1}{r^2}\left(1 - z - rz'\right), \tag{9}$$

and constitutes the first of our main equations. The second is obtained as a sum of Eqs. (2) and (3):

$$\kappa\left(\rho+p\right)=\frac{e^{-\lambda}}{r}\left(\nu'+\lambda'\right)=\frac{z}{r}\nu'-\frac{z'}{r}. \tag{10}$$

The third main equation will not be Eq.(4) but another combination of Eqs.(2)-(4) which varies from case to case. One can transform Eqs. (2)-(4) into expressions for p,q,ρ

$$2\kappa p=e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}-\frac{\lambda'}{2r}+\frac{3\nu'}{2r}+\frac{1}{r^2}\right)-\frac{1}{r^2}, \tag{11}$$

$$\frac{2q^2}{r^4}=e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}-\frac{\lambda'}{2r}-\frac{\nu'}{2r}-\frac{1}{r^2}\right)+\frac{1}{r^2}, \tag{12}$$

$$2\kappa\rho=e^{-\lambda}\left(-\frac{\nu''}{2}+\frac{\lambda'\nu'}{4}-\frac{\nu'^2}{4}+\frac{5\lambda'}{4r}+\frac{\nu'}{2r}-\frac{1}{r^2}\right)+\frac{1}{r^2}. \tag{13}$$

These equations may be written as linear first-order equations for z, suitable for the cases $(\nu,q)\,,(\nu,p)$ and (ν,ρ). Introducing $y=e^{\nu/2}$ we have

$$\left(r^2y'+ry\right)z'=-2\left(r^2y''-ry'-y\right)z-2y+\frac{4q^2}{r^2}y, \tag{14}$$

$$\left(r^2y'+5ry\right)z'=-2\left(r^2y''-ry'+y\right)z+2y-4\kappa\rho r^2y, \tag{15}$$

$$\left(r^2y'+ry\right)z'=-2\left(r^2y''+3ry'+y\right)z+2y+4\kappa pr^2y. \tag{16}$$

In the uncharged case the prescription of an equation of state makes the system of field equtions extremely difficult to solve. This is true even for the simplest realistic linear equation of state

$$p=n\rho-p_0, \tag{17}$$

where n is a parameter taking values in the interval $[0,1]$ for physically realistic solutions, while p_0 is a positive constant, allowing the existence of a boundary of the fluid where $p=0$. When $p_0=0$ we obtain the popular γ-law (with notation $n=\gamma-1$). In this case Eqs. (2)-(4) with $q=0$ lead to an Abel differential equation of the second kind independently from the approach or the coordinate system [46]. It is soluble in few simple cases. Almost all of the known few solutions of the more general Eq. (17) may be obtained by imposing a simple ansatz on λ which makes the system overdetermined [47]. Therefore, it is surprising that in the more complex charged case fluids, satisfying Eq. (17), are subjected to a linear equation, similar to Eqs. (14)-(16). Plugging Eq. (17) into Eq. (10) yields

$$\kappa\rho y=\frac{2}{n+1}\left[\frac{zy'}{r}-2\left(\frac{z'}{r}-\kappa p_0\right)y\right]. \tag{18}$$

Replacing this result in Eq. (15) leads to

$$\left(r^2y'+\frac{5n+1}{n+1}ry\right)z'=-2\left(r^2y''+\frac{3-n}{n+1}ry'+y\right)z+2y-\frac{4\kappa p_0}{n+1}r^2y. \tag{19}$$

We call this case (ν,n). It is a hybrid between (ν,ρ) and (ν,p) because fixing the equation of state, we fix the pressure in terms of the density. Eq. (17) is a privileged one, due to its linearity. Another realistic equation of state, namely the polytropic one, reads $p=n\rho^{1+1/k}$. Replaced in Eq. (10) it gives the relation

$$\kappa\left(\rho+n\rho^{1+1/k}\right)=\frac{2zy'}{ry}-\frac{z'}{r}. \tag{20}$$

This causes the appearance of radicals in Eq. (15) at best, and to non-integrable equations.

Eqs. (14)-(16) and (19) may be written in the general form

$$gz' = f_1 z + f_0, \tag{21}$$

whose quadrature is

$$z = e^F \left(C + H\right), \tag{22}$$

$$F = \int \frac{f_1}{g} dr, \qquad H = \int e^{-F} \frac{f_0}{g} dr. \tag{23}$$

Here and in the following C will denote a generic integration constant. They may be written also as linear second-order differential equations for y, useful in the cases $(\lambda, q), (\lambda, \rho), (\lambda, p)$ and (λ, n):

$$2r^2 z y'' + \left(r^2 z' - 2rz\right) y' + \left(rz' - 2z + 2 - \frac{4q^2}{r^2}\right) y = 0, \tag{24}$$

$$2r^2 z y'' + \left(r^2 z' - 2rz\right) y' + \left(5rz' + 2z - 2 + 4\kappa \rho r^2\right) y = 0, \tag{25}$$

$$2r^2 z y'' + \left(r^2 z' + 6rz\right) y' + \left(rz' + 2z - 2 - 4\kappa p r^2\right) y = 0, \tag{26}$$

$$2r^2 z y'' + \left(r^2 z' + 2\frac{3-n}{n+1} rz\right) y' + \left(\frac{5n+1}{n+1} rz' + 2z - 2 + \frac{4\kappa p_0}{n+1} r^2\right) y = 0. \tag{27}$$

The coefficient before the second derivative is one and the same in all cases. Eq. (24) is the generalization to the charged case [35,37] of the Wyman equation [38]. Obviously, the case $n = -1$ is not covered by Eqs. (19), (27).

One can find first-order differential equations also for the cases $(\lambda, *)$, based on the well-known Tolman-Oppenheimer-Volkoff (TOV) equation [42,48], generalized to the charged case [28]. Let us first derive briefly TOV. Eq. (3) may be written as an expression of ν' in terms of p, q and M

$$\nu' = \frac{2M + \kappa p r^3 - q^2/r}{r\left(r - 2M\right)}. \tag{28}$$

Now, let us take the following combination of equations $(3)' + \frac{2}{r}(3) - \frac{2}{r}(4)$. After some cancellations it gives

$$\kappa p' = -\kappa\left(\rho + p\right)\frac{\nu'}{2} + \frac{\left(q^2\right)'}{r^4}. \tag{29}$$

Inserting Eq. (28) into Eq. (29) yields the generalized TOV equation. We can trade q in Eqs.(28)-(29) for ρ and λ by using Eq. (9). The result is a Riccati equation for $Y \equiv \kappa\left(\rho + p\right)$

$$Y' = -\frac{r}{2} e^{\lambda} Y^2 + \frac{\lambda'}{2} Y - 4\kappa \frac{\rho}{r} + \frac{2}{r^4}\left(r^2 M'\right)'. \tag{30}$$

It marks another way to solve the cases $(\lambda, \rho), (\lambda, n = -1)$. Its solution yields for the pressure $\kappa p = Y - \kappa\rho$. Then ν is found from Eq. (10), while q is found from Eq. (9). Eq. (30) is a non-linear, but first-order companion of Eq. (15). Using Eq. (9) we find again a Riccati equation for the case (λ, q)

$$Y' = -\frac{r}{2} e^{\lambda} Y^2 + \frac{\lambda'}{2} Y + \frac{4q^2}{r^5} - \frac{8M'}{r^3} + \frac{2}{r^4}\left(r^2 M'\right)'. \tag{31}$$

From the definition of Y equations for the cases (λ, p) and (λ, n) follow

$$Y' = -\frac{r}{2} e^{\lambda} Y^2 + \left(\frac{\lambda'}{2} - \frac{4}{r}\right) Y + 4\kappa \frac{p}{r} + \frac{2}{r^4}\left(r^2 M'\right)', \tag{32}$$

$$Y' = -\frac{r}{2} e^{\lambda} Y^2 + \left(\frac{\lambda'}{2} - \frac{4}{\left(n+1\right) r}\right) Y - \frac{4\kappa p_0}{\left(n+1\right) r} + \frac{2}{r^4}\left(r^2 M'\right)'. \tag{33}$$

In all cases Riccati equations are obtained with the same coefficients before Y' and Y^2. They may be written as

$$Y' = -\frac{r}{2}e^{\lambda}Y^2 + f_2 Y + f_3, \tag{34}$$

where f_2 and f_3 are functions of r, differing from case to case. An arbitrary Riccati equation may be transformed into a linear second-order equation. The linearization of Eq. (34) is done by the change of variables $u = e^{\lambda/2}y$ and reads

$$u'' - \left(\frac{1}{r} + \lambda' + f_2\right)u' - \frac{r}{2}e^{\lambda}f_3 u = 0. \tag{35}$$

At this point f_3 contains λ'' which does not cancel. When a passage from u to y is done, λ'' cancels and we obtain exactly Eqs. (24)-(27). In this process we have exchanged Y, which is the sum of the pressure and the density, for y which is part of the metric.

So far we have reformulated the original system of Eqs. (2)-(4) into Eqs. (9), (10) and a third equation, presented in many different forms, adapted to the cases of the proposed classification. We have briefly discussed the cases (λ,ν), $(\lambda,*)$, $(\nu,*)$. The three remaining cases (ρ,p), (ρ,q) and (p,q) are the most natural ones since one prescribes two of the fluid characteristics, hoping that the third one and the metric will be regular and reasonable. The case (ρ,q) is easily reduced to (λ,q) because of Eq. (9). In the case (ρ,p), Y is also known and Eq. (30) becomes an equation for M

$$\frac{2}{r^4}\left(r^2M'\right)' + \frac{rM' - M}{r\,(r-2M)}Y - \frac{r^2Y^2}{2\,(r-2M)} - Y' - \frac{4\kappa\rho}{r} = 0. \tag{36}$$

This is an intricate, non-linear, second-order equation, which is not simpler than the TOV equation. It seems that it can be dealt with only numerically.

The third main equation for the case (p,q) is obtained in the following way. Let us replace the density in the TOV equation (29) with its expression from Eq. (9). After some tedious manipulations, the following equation for M is found

$$(M + g_0)\,M' = f_4 M + f_5, \tag{37}$$

$$g_0 = \frac{\kappa}{2}pr^3 - \frac{q^2}{2r}, \tag{38}$$

$$f_4 = \kappa r^3 p' - \frac{\left(q^2\right)'}{r} - \frac{\kappa}{2}r^2 p + \frac{q^2}{2r^2}, \tag{39}$$

$$2f_5 = -r^4\left(\kappa p' - \frac{\left(q^2\right)'}{r^4}\right) - r^2\left(\kappa p - \frac{q^2}{r^4}\right)\left(\frac{\kappa}{2}pr^3 - \frac{q^2}{2r}\right). \tag{40}$$

There is a standard procedure for the solution of such equations [46,49]. It consists of two changes of variables which bring them to the Abel equation of the second kind

$$\omega\omega_\zeta - \omega = f(\zeta), \tag{41}$$

where $\omega = M + g_0$ while

$$\zeta = g_0 + \int f_4 dr, \tag{42}$$

$$f(\zeta) = \frac{f_5 - g_0 f_4}{g_0' + f_4}. \tag{43}$$

Its integrable cases are few, depend on the shape of $f(\zeta)$ and are tabulated in Ref. [49].

As a whole, the most attractive are the mixed cases $(\nu, *)$ where one fluid characteristic and one metric component are specified. They lead to the most simple Eqs. (14)-(16) and (19).

The five functions which describe the fluid together with its gravitational field should satisfy some physical requirements. Eqs.(7)-(8) show that at the centre $M(0) = 0$ and $e^{\lambda} = 1$. The density and pressure should be positive and monotonously decreasing towards the boundary. It is obvious that q^2 should be positive, too. The boundary r_0 of the fluid sphere is determined by the relation $p(r_0) = 0$ where a junction to the Reissner-Nordström (RN) metric

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r} + \frac{e^2}{r^2} \tag{44}$$

should be performed. The metric and ν' must be continuous there. This leads to the expressions

$$\frac{m}{r_0} = 1 - e^{\nu}\left(1 + \frac{r_0 \nu'}{2}\right) = \frac{M(r_0)}{r_0} + \frac{q^2(r_0)}{2r_0^2}, \tag{45}$$

$$\frac{e^2}{r_0^2} = 1 - e^{\nu}(1 + r_0 \nu') = \frac{q^2(r_0)}{r_0^2}. \tag{46}$$

The condition $e = q(r_0)$ follows from the vanishing of the pressure and vice versa. In fact, Eq. (10) gives at r_0

$$z' = z\nu' - \kappa r_0 \rho(r_0). \tag{47}$$

Replacing this in Eq. (9) and taking into account that $e^{\nu}(r_0) = e^{-\lambda}(r_0)$, we get a proof of the assertion.

Solutions, satisfying the above conditions are called in the following physically realistic. This is done in agreement with the majority of the reviewed papers, and their satisfaction is already rather non-trivial. Some other, more stringent requirements, are discussed in Refs. [50–52]. The most important of them is $0 \leq \frac{dp}{d\rho} \leq 1$, which means that the speed of sound is positive and causal, i.e., not greater than the speed of light. One has control over this characteristic in CD and models with linear equation of state. The case (ν, q), which is a generalization of uncharged solutions, possesses their behavior for small q. In other cases the expressions for density and pressure may be so complicated that the fulfilment of this requirement is hard to estimate. We do not discuss these additional conditions in order to contain the present paper in a reasonable volume and prevent its dissociation into a set of separate publications. In many cases the examples have free parameters, or such can be added easily, so additional fine-tuning may be done.

III. THE CASES (λ, ν) AND CONSTANT T_0^0

In the case (λ, ν) Eqs. (11)-(13) should be used to find q, p and ρ. This is the simplest case but control over pressure and density is completely lost and one must proceed by trial and error. Krori and Barua [33] have given the solution

$$\lambda = a_1 r^2, \tag{48}$$

$$\nu = a_2 r^2 + a_3, \tag{49}$$

where a_i denote generic constants of a known solution. They are fixed by the junction conditions. The solution is non-singular and the positivity conditions are satisfied.

Eqs.(7)-(8) show that the generalized Schwarzschild condition $T_0^0 = const$ determines λ

$$e^{-\lambda} = 1 - ar^2, \tag{50}$$

where a is a positive constant. We have the freedom to choose one more function. In Ref. [6] the condition $p = 0$ was further imposed. We shall review this solution in Sec.V. In Ref. [21] some relations between ϕ and ν were utilized. They can be arbitrary in the perfect fluid case, so we do not base the classification on such criteria. One should use instead Eqs. (24)-(27) or Eqs. (30)-(33). The simplest expressions are obtained in the case (λ, Y) when Eq. (30) is used to determine ρ

$$4\kappa\rho = -rY' - \frac{r^2 Y^2}{2(1 - ar^2)} + \frac{arY}{1 - ar^2} + 12a. \tag{51}$$

Then $\kappa p = Y - \kappa\rho$ and

$$q^2 = r^4 \left(3a - \kappa\rho\right) = \frac{r^5}{4}\left[Y' + \frac{rY\left(Y-2a\right)}{2\left(1-ar^2\right)}\right]. \tag{52}$$

The function ν is determined by a simple integration from Eq. (10)

$$\nu' = \frac{r}{1-ar^2}\left(Y - 2a\right). \tag{53}$$

This is an advantage over the second-order equations (24)-(27). Eqs. (50)-(53) solve the problem in a minimal algebraic way. Several positivity conditions follow for the master function Y and ρ: $3a > \kappa\rho, Y > 2a, Y > \kappa\rho, Y' < 0$.

In order to illustrate how the above scheme works one may take any of the solutions elaborated in Ref. [21], extract Y and check how the above equations and inequalities are satisfied. The simplest case has

$$Y = 2a + a_1\left(1-ar^2\right)^{1/2}. \tag{54}$$

Of course, in any of the cases $(\lambda, *)$ we can study the subcase given by Eq. (50). This will be done in the following sections.

A similar problem arises when ν is given. Which function should be prescribed in addition in order to have the simplest algorithm for generating solutions? Again, Y is the best choice. This is seen by taking the difference of Eqs. (25),(26) or reformulating Eq. (10)

$$yz' - 2y'z + ryY = 0. \tag{55}$$

This equation is much simpler than any of Eqs. (14)-(16).

IV. CHARGED DE SITTER SOLUTIONS AND THEIR GENERALIZATION

In the uncharged case this is a special integrable case with linear equation of state (17) and $n = -1$, $p_0 = 0$ which is equivalent to the de Sitter solution. The charged case is also completely solvable and falls in the case $(\lambda, Y = 0)$ according to our classification. The third main equation is Eq. (30) which becomes

$$\kappa\rho = \frac{\left(r^2 M'\right)'}{2r^3} = \frac{1}{2r^2}\left(2M' + rM''\right). \tag{56}$$

Eq. (10) gives $\nu = -\lambda$ which is a feature also of the exterior RN solution. Eq. (9) yields

$$q^2 = -\frac{r^5}{2}\left(\frac{M'}{r^2}\right)' = \frac{r}{2}\left(2M' - rM''\right). \tag{57}$$

Thus, when M is given, all other unknowns follow from simple formulae. Eq. (57) allows also one to take q as a basis:

$$\kappa\rho = 2a_0 - \frac{q^2}{r^4} - 4\int_0^r \frac{q^2}{r^5}dr, \tag{58}$$

$$M = -2\int_0^r \bar{r}^2 \int_0^{\bar{r}} \frac{q^2}{\check{r}^5} d\check{r} d\bar{r} + \frac{a_0}{3}r^3. \tag{59}$$

Here a_0 is some positive constant and clearly $q = r^{2+\varepsilon} q_0(r)$ where $\varepsilon > 0$ and $q_0(0) = const$. Eqs. (58)-(59) demonstrate the process of 'electrification' of de Sitter space. The bigger the charge function, the lower the density until some point r_0 is reached where $\rho(r_0) = 0$ and consequently the fluid sphere acquires a boundary. We already know that $M > 0$. Eqs. (56)-(57) show that ρ and q^2 are both positive only when $M' > 0$ and $2M' \geq r\left|M''\right|$. The equality holds at the boundary, where $M''(r_0) < 0$. The charge and the mass of the solution follow from the junction conditions (45),(46)

$$e^2 = 2r_0^2 M'(r_0), \tag{60}$$

$$m = M(r_0) + r_0 M'(r_0). \tag{61}$$

Obviously, there is an abundance of solutions since M and its first two derivatives have to satisfy few simple inequalities. Three solutions are known in the literature. The $T_0^0 = const$ condition leads to $q = 0$ in the interior and consequently to the de Sitter solution, which has constant density. One can introduce, however, a surface charge $\sigma \sim \delta(r - r_0)$ which gives $q = \theta(r - r_0)\, e$, $\rho = \rho_0$ and

$$z = 1 - \frac{\kappa}{3}\rho_0 r^2. \tag{62}$$

This is the solution of Cohen and Cohen [27,29]. Another solution [28] has

$$M = \frac{\kappa^2}{360}\sigma_0^2 r^3 \left(5a_1^2 - 2r^2\right), \tag{63}$$

$$q = \frac{\kappa}{6}\sigma_0 r^3, \tag{64}$$

$$\rho = \frac{\kappa}{12}\sigma_0^2 \left(a_1^2 - r^2\right), \tag{65}$$

$$z = 1 - \frac{\kappa^2}{36}\sigma_0^2 a_1^2 r^2 + \frac{\kappa^2}{90}\sigma_0^2 r^4. \tag{66}$$

Here σ_0 enters the charge density $\sigma = \sigma_0 e^{-\lambda/2}$, while a_1 is related to $a_0 = \frac{\kappa^2}{24}\sigma_0^2 a_1$. This is one of the electromagnetic models of the electron. When $\sigma_0 \to 0$ we have $\rho, p, q, \lambda, \nu \to 0$ due to $a_0 \to 0$. The limit is flat spacetime. We have shown, however, that a_0 is not obliged to vanish when $q = 0$ so that, in general, the case $(\lambda, Y = 0)$ is an electric generalization of de Sitter spacetime. The natural generalization of flat spacetime is the CD solution, which will be discussed in the next section.

A third solution has been found by Gautreau [30]. It has

$$2M = c_1^2 r - \frac{c_1^2 a_2^2}{r}\sin^2\frac{r}{a_2}, \tag{67}$$

$$z = 1 - c_1^2 + \left(\frac{c_1 a_2}{r}\sin\frac{r}{a_2}\right)^2, \tag{68}$$

$$\kappa\rho = \frac{c_1^2}{r^2}\sin^2\frac{r}{a_2}, \tag{69}$$

$$\phi = c_1 + \frac{c_1 a_2}{r}\sin\frac{r}{a_2}, \tag{70}$$

and $q = -r^2\phi'$. Many other solutions are possible and we give one simple example, a variation of the solution in Ref. [28]:

$$M = a\left(r^3 - r^4\right), \tag{71}$$

$$\kappa\rho = 2a(3 - 5r), \tag{72}$$

$$q^2 = 2ar^5, \tag{73}$$

$$z = 1 - 2ar^2 + 2ar^3. \tag{74}$$

The junction at $r_0 = 3/5$ gives $m = ar_0^3$, $e^2 = 2ar_0^5$. M satisfies all necessary inequalities and the density is positive and decreasing.

The case $Y = 0$ is easy because Eq. (30) collapses into the simple relation (56). A generalization can be made by taking Eq.(56) as a basis. This constitutes the special case $(\lambda, \rho(\lambda))$. Then q is given again by Eq. (57), while Eq. (30) becomes

$$Y' = -\frac{r}{2}e^{\lambda}Y^2 + \frac{\lambda'}{2}Y. \tag{75}$$

This is a Bernoulli equation and, unlike the Riccati equation, it is readily soluble in quadratures. Its general solution gives an expression for the pressure

$$\kappa p = \frac{e^{\lambda/2}}{C + \frac{1}{2}\int e^{3\lambda/2} r dr} - \kappa\rho, \tag{76}$$

where $C^{-1} = Y(0)$. When $C \to \infty$ we return to the previous case $Y = 0$, the trivial solution of Eq. (75). Fortunately, Eq. (10) can be integrated explicitly too and a closed expression is obtained for ν

$$e^{\nu/2} = A^{-1}e^{-\lambda/2}\left(1 + \frac{1}{2C}\int_0^r e^{3\lambda/2} r dr\right), \tag{77}$$

$$A = 1 + \frac{1}{2C}\int_0^{r_0} e^{3\lambda/2} r dr. \tag{78}$$

The second equation follows from the junction conditions. Eqs. (56),(57),(76)-(78) provide the generalization of the case $Y = 0$. Thus, every function M leads to two solutions: one with trivial Y and one with non-trivial Y, satisfying Eq. (75). The trivial solution has the disadvantage of negative pressure. Eq.(76) suggests that in the non-trivial case solutions with positive pressure may exist.

Let us take Eq. (63) with somewhat different constants

$$M = br^3\left(1 - br^2\right). \tag{79}$$

Then

$$\kappa\rho = 3b\left(2 - 5bx\right), \tag{80}$$

$$q^2 = 5b^2x^3, \tag{81}$$

$$\kappa p = \frac{1}{C\left(1 - 2bx + 2bx^2\right)^{1/2} - \frac{1}{4b} + \frac{x}{2}} + 15b^2x - 6b, \tag{82}$$

where $x = r^2$ and $b > 0$. When $x < 2/5b$ the density is positive and decreasing. The pressure is positive at the centre when $1/4b < C < 5/12b$. Let us choose $b = 0.01$ and $C = 40$. Then p has a maximum at $x_0 = 0.3$ and a root at $x_0 = 2.5$. This is a semi-realistic interior solution with $e^2 = 5b^2x_0^3$ and mass given by Eq. (61).

V. THE CASE OF GIVEN PRESSURE. CHARGED DUST

In this section we discuss the cases (ν, p) and (λ, p). The pressure is considered a known positive function which decreases monotonously outwards and vanishes at the boundary of the fluid sphere. The simplest case is $p = 0$ (the only reasonable case with constant pressure, unlike the cases with constant density). This represents charged dust. The case (ν, p) is much easier since the third main equation (16) is linear and first-order. Something more, in Eq. (21) $f_1 = -2g'$. This allows to obtain a compact expression for z

$$z = \frac{1}{\left(1 + \frac{r\nu'}{2}\right)^2} + \frac{C}{r^2e^{\nu}\left(1 + \frac{r\nu'}{2}\right)^2} + \frac{4\kappa}{r^2e^{\nu}\left(1 + \frac{r\nu'}{2}\right)^2}\int_0^r\left(1 + \frac{r\nu'}{2}\right)e^{\nu}r^3 p dr. \tag{83}$$

The knowledge of ν allows to satisfy two of the junction conditions by choosing a function continuous at r_0 together with its derivative. Eqs.(9)-(10) provide an expression for q in terms of the known ν, p and with $z(\nu, p)$ from Eq. (83)

$$\frac{q^2}{r^2} = 1 - z\left(1 + r\nu'\right) + \kappa p r^2. \tag{84}$$

The density follows from Eq. (10)

$$\kappa\rho = \frac{z}{r}\left(\nu' - \frac{z'}{z}\right) - \kappa p. \tag{85}$$

Its structure is clarified when z' is replaced by ν', using Eq. (83)

$$\kappa\rho = \frac{2z}{r^2} - \frac{2}{r^2\left(1+\frac{r\nu'}{2}\right)} + \frac{2z\nu'}{r} + \frac{z\nu'}{r\left(1+\frac{r\nu'}{2}\right)} + \frac{z\nu''}{1+\frac{r\nu'}{2}} - \frac{4\kappa p}{1+\frac{r\nu'}{2}} - \kappa p. \tag{86}$$

When $r \to 0$, $z \to 1$ and the third term on the right produces a pole unless $\nu' \to 2\nu_0 r$, which means $\nu'' \to 2\nu_0$. Then $r\nu' \to 0$ and the poles in the first two terms cancel. The last two terms approach negative constants. In order to compensate them and have a positive density at the centre, ν_0 must be a positive constant and the inequality $8\nu_0 > 5\kappa p(0)$ should hold. Another consequence is that e^{ν} is an increasing function in the vicinity of the centre. Since $e^{-\lambda}$ is a decreasing function which starts from 1 and equals e^{ν} at r_0, we have $e^{\nu}(0) < 1$. Eq. (83) shows that z has a pole at $r = 0$ unless $C = 0$.

Now we can understand the physical meaning of the terms in z. The third term represents the contribution from the pressure to the metric. When $p = 0$, z is still non-trivial and represents the general CD solution. It has a pole at the beginning of the coordinates and should not be used there. When $C = 0$ only the first term remains and this is the regular CD solution in curvature coordinates [1]. The intricate proofs that this is the most general regular solution [2,4] are replaced here by the obvious condition $C = 0$. The second term in Eq. (83) may be induced in principle by the third if the lower limit of the integral is changed. The first term may be absorbed by the third if one uses the identity

$$\left(1 + \frac{r\nu'}{2}\right) e^{\nu} r = \frac{1}{2}\left(r^2 e^{\nu}\right)', \tag{87}$$

and makes the shift $\kappa p \to \kappa p + \frac{1}{2r^2}$. The expression under the integral in Eq. (83) is bell-shaped, since it vanishes both at zero and r_0 and is positive in-between.

Let us discuss in detail first the case of regular CD. It provides an excellent illustration of the classification scheme, adopted in the present paper. In the case $(\nu, p = 0)$ Eqs. (5)-(6) and (83)-(86) give

$$e^{\lambda} = \left(1 + \frac{r\nu'}{2}\right)^2, \tag{88}$$

$$q = \frac{r^2\nu'}{2\left(1+\frac{r\nu'}{2}\right)}, \tag{89}$$

$$\kappa\rho = \kappa\sigma = \frac{r\nu'' + 2\nu' + \frac{1}{2}r\nu'^2}{r\left(1+\frac{r\nu'}{2}\right)^3}, \tag{90}$$

$$\phi = -e^{\nu/2} + \phi_0. \tag{91}$$

For the case $(\lambda, p = 0)$ we should use Eq. (88) to express everything via λ

$$\nu' = \frac{2}{r}\left(e^{\lambda/2} - 1\right), \tag{92}$$

$$q = r\left(1 - e^{-\lambda/2}\right), \tag{93}$$

$$\kappa\rho = \kappa\sigma = \frac{2}{r^2}\left(e^{-\lambda/2} - e^{-\lambda} + \frac{r}{2}\lambda' e^{-\lambda}\right). \tag{94}$$

These formulae involve only the first derivative of λ. The case $(q, p = 0)$ is explicitly solvable, unlike the general case (q, p):

$$e^{\lambda} = \frac{r^2}{(r-q)^2}, \tag{95}$$

$$M = q - \frac{q^2}{2r}, \tag{96}$$

$$\nu' = \frac{2q}{r\,(r-q)}, \tag{97}$$

$$\kappa\rho = \kappa\sigma = \frac{2}{r^3}\,(r-q)\,q'. \tag{98}$$

The condition $q < r$ must hold. These formulae clearly demonstrate the electrification of flat-space, which is the trivial dust solution in the uncharged case. If we put $p = 0$ in Eqs. (37)-(40) we still obtain a complicated Abel equation. One can check that M, given by Eq. (96), satisfies it. The most complicated is the $(\rho, p = 0)$ case. If we introduce w via

$$e^{\lambda} = w^{-2}, \tag{99}$$

then Eqs. (92),(93) give

$$\nu' = \frac{2\,(1-w)}{rw}, \tag{100}$$

$$q = r\,(1-w)\,, \tag{101}$$

while Eq. (94) becomes an equation for w

$$rww' = -w^2 + w - \frac{\kappa}{2}r^2\rho. \tag{102}$$

This is an Abel differential equation of the second kind like Eq. (36). The procedure for its solution was described briefly after Eq. (40) and brings it to the canonical form [46,49]

$$WW' - W = -\frac{\kappa}{2}r^3\rho, \tag{103}$$

where $W = wr$. The set of density profiles, leading to integrable W is very restricted.

Several explicit dust solutions in curvature coordinates have been given. Efinger [5] studied the simple case $e^{-\lambda} = 4/9$ in the presence of a cosmological constant. When it is zero we have $q = r/3$ and singular $e^{\nu} = a_1 r$, $\rho = \sigma = 4/9\kappa r^2$, $F_{10} = \frac{1}{2}\sqrt{a_1}\,r^{-1/2}$. Florides illustrated his general discussion [4] with a power-law solution $q = a_2 r^{a_3+3}$,

$$e^{\lambda} = \left(1 - a_2 r^{a_3+2}\right)^{-2}, \tag{104}$$

$$e^{\nu} = \left(1 - a_2 r^{a_3+2}\right)^{-\frac{2}{a_3+2}}, \tag{105}$$

$$\rho = \sigma = \sigma_0\left(\frac{r}{b_1}\right)^{a_3} e^{-\lambda/2}, \tag{106}$$

where $a_2 = \frac{\kappa\sigma_0}{2(a_3+3)b_1^{a_3}}$. Finally, in Ref. [6] the case of constant T_0^0 was discussed. It leads to Eq. (50) and Eqs. (92)-(94) give

$$\nu = 2\ln\left[a_4 \frac{1-\left(1-a^2r^2\right)^{1/2}}{a^2r^2}\right], \tag{107}$$

$$q^2 = r^2\left[2-a^2r^2-2\left(1-a^2r^2\right)^{1/2}\right], \tag{108}$$

$$\kappa\rho = 4a^2 - \frac{2}{r^2}\left[1-\left(1-a^2r^2\right)^{1/2}\right]. \tag{109}$$

We use the known solutions just to illustrate the general ideas and formulae and do not elaborate on their physical significance or details of the junction conditions, which may be found in the original references.

As mentioned in the introduction, most work on CD has been done in isotropic coordinates

$$ds^2 = U^{-2}dt^2 - U^2\left(dr^2 + r^2 d\Omega^2\right), \tag{110}$$

where the simple relation $\phi = \pm U^{-1}$ holds and the only equation to be solved is

$$U'' + \frac{2}{r}U' = -\frac{\kappa}{2}U^3\rho. \tag{111}$$

When U is given, we have the mixed case $(U, p = 0)$ and the density is readily determined from Eq. (111). The function $U^{-2} = a_1r^2 + a_2$ was used in Refs. [10,15]. Another, more general function

$$U = 1 + \frac{a_3}{r_0} + \frac{a_3\left(r_0^k - r^k\right)}{kr_0^{k+1}} \tag{112}$$

was studied for $k = 2$ in Ref. [12], for $k = 4$ in Ref. [11] and for general k in Ref. [13]. Recently, the function $U = a_4\frac{\sin a_5 r}{r}$ was studied [16,17].

When ρ is given, it is convenient to transform Eq. (111) into

$$U_{XX} = -\frac{\kappa\rho}{2}\frac{U^3}{X^4}, \tag{113}$$

where $X = 1/r$. We may consider $\rho(U)$ as a known function, chosen to simplify this equation. One possibility, leading to Bessel functions, is $\rho = a_6/U^2$ [16,17]. Other choices lead to two-dimensional integrable models, like the sine-Gordon equation [18]. The metric (110) describes also the regular static CD. Then the case of constant fluid density ρ_0 is soluble in cn, one of the Jacobi elliptic functions, but the solution cannot be spherically symmetric. In the spherically symmetric case Eq. (113) becomes an Emden-Fowler equation, whose integrable cases are tabulated in Ref. [49]. When the power of U is 3 like in Eq. (113), the integrable cases have X^3 or X^6 and not X^4. We can make the transformation $\psi = -U/r$, $Z = U_r - U/r$ and obtain

$$ZZ_\psi - Z = -\frac{\kappa\rho_0}{2}\psi^3. \tag{114}$$

This equation coincides exactly with the Abel equation (103) in curvature coordinates with constant density and is non-integrable too.

Let us go back to curvature coordinates (1) which are more convenient when one wants to study also the case of non-vanishing pressure. From the considerations about the regularity of ρ, given after Eq. (86), it follows that the simplest function ν would be $\nu = \nu_0 r^2$. This choice coincides with Eq. (49) and leads to exponential functions and probably to the error function in Eq. (83). We shall choose a different option

$$e^\nu = a + br^2, \tag{115}$$

where $0 < a < 1$ and $b > 0$. This ansatz has been thoroughly studied in the uncharged case [50,52]. The CD solution build on it has

$$z = \left(\frac{a+br^2}{a+2br^2}\right)^2, \tag{116}$$

$$q = \frac{br^3}{a + 2br^2}, \tag{117}$$

$$\kappa\rho = \frac{2b\left(a + br^2\right)\left(3a + 2br^2\right)}{\left(a + 2br^2\right)^3}. \tag{118}$$

Density is monotonously decreasing. The junction conditions (45)-(46) give

$$e = \frac{br_0^3}{a + 2br_0^2}, \tag{119}$$

$$m = r_0\left(1 - a - 2br_0^2\right). \tag{120}$$

The condition $z = e^\nu$ at r_0 supplies the relation

$$r_0^2 = \frac{1 - 4a + \sqrt{1 + 8a}}{8b}, \tag{121}$$

which ensures the obligatory $e = m$. The r.h.s. is positive when $a < 1$, which is the case.

The main disadvantage of regular CD solutions is that they possess a fixed ratio e/m which is unrealistic, especially for classical electron models, and requires the extreme RN solution as an exterior. Eq. (83) tells that when the point $r = 0$ is excluded, general CD solutions are possible. They have the following characteristics

$$z = \left(1 + \frac{r\nu'}{2}\right)^{-2}\left(1 + C\frac{e^{-\nu}}{r^2}\right), \tag{122}$$

$$\kappa\rho = \left(1 + \frac{r\nu'}{2}\right)^{-2}\left(1 + C\frac{e^{-\nu}}{r^2}\right)\frac{\nu'}{r} - \frac{z'}{r}, \tag{123}$$

$$\frac{q^2}{r^2} = 1 - \frac{1 + r\nu'}{\left(1 + \frac{r\nu'}{2}\right)^2}\left(1 + C\frac{e^{-\nu}}{r^2}\right), \tag{124}$$

$$\phi' = -e^{\nu/2}\frac{\left[r^4\nu'^2 - 4C\left(1 + r\nu'\right)e^{-\nu}\right]^{1/2}}{2r\left(r^2 + Ce^{-\nu}\right)^{1/2}}. \tag{125}$$

When $C = 0$ they reduce to Eqs. (88)-(91). What is the physical significance of such solutions? Let us consider a core of perfect fluid, occupying a ball with radius r_0, with given ν and p. Then z is determined from Eq. (83) with $C = 0$:

$$z = \frac{1}{\left(1 + \frac{r\nu'}{2}\right)^2}\left[1 + \frac{C\left(r\right)}{r^2e^\nu}\right], \tag{126}$$

$$C\left(r\right) = 4\kappa\int_0^r\left(1 + \frac{r\nu'}{2}\right)e^\nu r^3 p dr. \tag{127}$$

At the junction z is equivalent to Eq. (122) where $C = C_0 \equiv C\left(r_0\right)$. An observer cannot understand whether the interior solution consists of perfect fluid or general CD since the only imprint left by the pressure distribution inside is a constant. In principle, the RN metric should be taken as an exterior, but there are no obstacles to take a general CD metric and postpone the junction to RN till another point $r_1 > r_0$. Thus we obtain a triple-layered model with a perfect fluid core up to r_0 (zone I), a halo of general CD from r_0 to r_1 (zone II) and the RN solution for $r > r_1$ (zone III). The metric component z is given correspondingly by Eq. (126), Eq. (122) with C_0 and Eq. (44). For this purpose we choose a continuous, together with its derivative, function ν in the region $0 < r < r_1$. At the first junction z is also continuous, while the pressure drops to zero. The density is finite. In zone II the pressure remains

zero, while the density continues to decrease. Finally, at r_1 the density also drops to zero (perhaps with a jump) and a RN solution follows till infinity, making the composite solution asymptotically flat. As seen from Eqs. (123),(124), the constant C_0 shifts e/m from 1, like a perfect fluid solution, occupying zones I and II would do.

This picture will be backed by an explicit example with the ansatz (115). Since the details should not depend on the form of $p(r)$ but only on the value of C, we shall discuss first zone II and its junction with zone III. In the interval $[r_0, r_1]$ we have

$$z = \frac{\left(a + br^2\right)\left[C_0 + r^2\left(a + br^2\right)\right]}{r^2\left(a + br^2\right)^2}, \tag{128}$$

while in general

$$\kappa\rho = \frac{z}{r}\left[\frac{2}{r} + \frac{8br}{a + 2br^2} - \frac{C'(r) + 2r\left(a + 2br^2\right)}{C(r) + ar^2 + br^4}\right]. \tag{129}$$

For CD $C' = 0$ and one can easily show that the density is positive for positive C_0. It decreases towards zero when $r \to \infty$. The charge function is given by Eqs. (115), (124). Its square is positive when

$$C_0 < \frac{b^2 r_0^6}{a + 3br_0^2}. \tag{130}$$

The total mass and charge are given by Eqs. (45),(46)

$$m = r_1\left(1 - a - 2br_1^2\right), \tag{131}$$

$$e^2 = r_1^2\left(1 - a - 3br_1^2\right). \tag{132}$$

The junction condition $-\lambda = \nu$ at r_1 gives

$$C_0 = m^2 - e^2. \tag{133}$$

This relation clearly shows the effect of pressure in zone I on the mass-charge ratio, not altered by the CD halo in zone II. The result in zone I was found in Ref. [6].

The mass and squared charge are positive when $1 - a > 3br_1^2$. The constant C_0 is positive when

$$4br_1^2\left(a + br_1^2\right) > a^2 - a + br_1^2. \tag{134}$$

A sufficient condition is

$$\frac{1}{4} < br_1^2 < \frac{1}{3}(1 - a), \tag{135}$$

which requires also $a < 0.24$. Let us take $a = 0.01$ and $br_1^2 = 0.3$. Then Eq. (130) is satisfied when $\left(\frac{r_0}{r_1}\right)^2 \geq 0.8$ and utilizing the equality we get $r_1 = 4.013\sqrt{C_0}$, $r_0 = 3.588\sqrt{C_0}$, $b = 0.075/\sqrt{C_0}$. This proves the existence of CD solutions with $|e|/m \neq 1$. They sustain the value of C, obtained in zone I from a perfect fluid solution with positive pressure.

It seems that there are no solutions of the type $(\nu, p > 0)$ in the literature. It must be stressed that this case is completely general, unlike the cases of constant T_0^0 or with $\rho + p = 0$, discussed in the previous sections. Every perfect fluid solution may be reformulated as a (ν, p) case. Let us proceed with the ansatz (115) which transforms Eq. (127) into

$$C(r) = 4\kappa\int_0^r \left(a + 2br^2\right) r^3 p\, dr. \tag{136}$$

We base the discussion on $C(r)$ as a fundamental object, hence, it is convenient to select a more involved p leading to a simple $C(r)$. Let us take

$$\kappa p = \frac{p_0 - p_1 r^2}{a + 2br^2} \tag{137}$$

and then

$$C(r) = r^4 \left(p_0 - \frac{2}{3} p_1 r^2\right), \tag{138}$$

$$z = \frac{\left(a + br^2\right)^2}{\left(a + 2br^2\right)^2} \left[1 + \frac{r^2 \left(p_0 - \frac{2}{3} p_1 r^2\right)}{a + br^2}\right], \tag{139}$$

$$\frac{q^2}{r^2} = \frac{r^4}{\left(a + 2br^2\right)^2} \left(b - bp_0 - \frac{1}{3} a p_1\right), \tag{140}$$

where z follows from Eq. (126) while q is calculated from the general expression (84). The density is given by Eq. (129). Here p_0 and p_1 are positive constants, connected by $p_0 = p_1 r_0^2$. The ansatz (115) yields $\nu_0 = b/a$ and the condition $\rho(0) > 0$ becomes $8b/5a > \kappa p(0) = p_0/a$. This is weaker than a necessary condition for positive q^2, $b > p_0$. At the boundary the density is always positive since $C'(r_0) = 0$ and Eq. (129) coincides with the expression for general CD. We have also $p_0 = 3C_0/r_0^4$ and using the numerical values of the CD example, one can express p_0 and p_1 in terms of C_0, in addition to b, r_0 and r_1. It can be shown that for $C > 0.06$, ρ and q^2 are positive and the solution is physically realistic. Thus we have constructed explicitly a triple solution of the type PF-CD-RN. This ends the discussion of case (ν, p).

The case (λ, p) relies on Eq. (26). Even the simplest constant z leads to special functions or non-integrable equations for y when p is chosen physically realistic.

VI. THE CASE (ν, Q)

The cases (ν, q) and (λ, q) comprise the most direct generalizations of the numerous uncharged solutions. Setting $q = 0$ one obtains one of them with the chosen ansatz for ν or λ. Like the previous section, the case (ν, q) is much simpler. The third main equation is Eq. (14). Now f_1 is not simply $-2g'$ but it can be represented as a linear combination of g', g and y which take care for y'', y' and y respectively. This is a general result which holds for any of Eqs. (14)-(16), (19). In this particular case we have

$$f_1 = -2g' + \frac{8g}{r} - 4y, \tag{141}$$

$$e^F = \frac{r^2 e^{2h}}{e^\nu \left(1 + \frac{r\nu'}{2}\right)^2}, \tag{142}$$

$$H = 2 \int e^\nu \left(1 + \frac{r\nu'}{2}\right) e^{-2h} r^{-3} \left(\frac{2q^2}{r^2} - 1\right) dr, \tag{143}$$

$$h = \int \frac{\nu' dr}{1 + \frac{r\nu'}{2}} \tag{144}$$

and z is given by Eq. (22). We shall use an ansatz, slightly more general than (115)

$$e^\nu = \left(a + br^2\right)^k, \tag{145}$$

where k is an integer. In the uncharged case $k = 1$ leads to Tolman's solution IV [42]. The case $k = 2$ was discussed first by Wyman [38] as an extension of Tolman's solution VI and later was studied in detail by Adler [39]. Solution with $k = 3$ was given by Heintzmann [53] (see also Ref. [52]). The general class was studied by Korkina [54], but her only explicit solution was the one of Heintzmann. Later Durgapal [55] studied in detail the cases $k = 1 - 5$. All of them satisfy the physical criteria used in this paper, but some have irregular behaviour of the speed of sound $dp/d\rho$ and p/ρ.

Going to the charged case, Eq. (145) and Eqs. (45), (46) give

$$e^2 = q^2(x_0), \tag{146}$$

$$\frac{m}{r_0} = 1 + \frac{1+(k+1)\tau x_0}{1+(2k+1)\tau x_0}\left(\frac{e^2}{x_0} - 1\right), \tag{147}$$

where $\tau = b/a$, $x = r^2$. Let us consider first the case $k = 1$. Then

$$z = \frac{1+\tau x}{1+2\tau x}\left(1 + Cx + 2xQ\right), \tag{148}$$

$$Q = \int_0^x \frac{q^2}{x^3} dx. \tag{149}$$

The pressure and the density are given by

$$\kappa\rho = \frac{\tau - 3C(1+\tau x)}{1+2\tau x} + \frac{2\tau(1+Cx)}{(1+2\tau x)^2} - \frac{2(1+\tau x)}{1+2\tau x}\left(3Q + \frac{2q^2}{x^2}\right) + \frac{4\tau x Q}{(1+2\tau x)^2}, \tag{150}$$

$$\kappa p = \frac{\tau + C + 3\tau C x + 2(1+3\tau x)Q}{1+2\tau x} + \frac{q^2}{x^2}. \tag{151}$$

When $q = 0$ these expressions coincide with the pressure and the density from Refs. [42,55]. We have $\kappa\rho(0) = 3(\tau - C)$. Let us choose $q^2 = K^2 x^3$. Then the condition $p(x_0) = 0$ yields an expression for C

$$C = -\frac{\tau + (3+8\tau x_0)K^2 x_0}{1+3\tau x_0}, \tag{152}$$

which means that C is negative and consequently $\rho(0)$ is positive. The condition $z = e^{\nu}$ at x_0 defines a as

$$a = \frac{1 + Cx_0 + 2K^2 x_0^2}{1+2\tau x_0}. \tag{153}$$

Together with Eq. (146) which becomes $e^2 = K^2 x_0^3$ and Eq. (147) for $k = 1$ we can express c, a, e, m through x_0, τ, K.

Let us consider next the case $k = 2$. Then Eqs. (142)-(145) give

$$e^F = \frac{x}{(a+3bx)^{2/3}}, \tag{154}$$

$$z = 1 + Ce^F + e^F \int \frac{2(a+bx)q^2 dx}{(a+3bx)^{1/3} x^3}. \tag{155}$$

When $q = 0$ this is the result of Adler who used the simplifying assumption $e^F H = 1$ instead of the ansatz (145). When $q^2 = K^2 x^3$ we obtain

$$z = 1 + \frac{Cx}{(a+3bx)^{2/3}} + \frac{4a}{5b}K^2 x + \frac{2}{5}K^2 x^2. \tag{156}$$

This is the corrected result of Nduka [36] who wrote the field equations with an error in the sign of q^2. In all his results K^2 should be replaced by $-K^2$ and the conclusions correspondingly altered.

In another paper Singh and Yadav [37] simplified Eqs. (14), (21) by demanding $f_1 = -f_0$. This leads to the following equation for y

$$r^2 y'' - ry' - \frac{2q^2}{r^2} y = 0. \tag{157}$$

Letting $q = Kr$ they obtain the Euler equation. It has three types of solutions, all of which are singular at $r = 0$ since $y \sim r^{a_1}$. If $K = 0$ the non-singular solution of Adler is again recovered.

VII. THE CASE (ν, N)

This is the case with given ν and the perfect fluid satisfying the linear equation of state Eq. (17). Realistic solutions have $0 < n \leq 1$ which means constant speed of sound and causal behavior. Surprisingly, the charged case is much easier than the uncharged one. The third main equation is Eq. (19) with $n \neq -1$. An expression for f_1, analogous to Eq. (141), may be written

$$f_1 = -2g' + \frac{16ng}{(n+1)\,r} - \frac{8n\,(9n+1)}{(n+1)^2}y. \tag{158}$$

Then F and H read

$$e^F = \frac{r^{-(\alpha+1)}e^{\beta h_1}}{e^\nu\left(A + \frac{r\nu'}{2}\right)^2}, \tag{159}$$

$$H = 2\int e^\nu \left(A + \frac{r\nu'}{2}\right) r^\alpha e^{-\beta h_1}\left(1 - \frac{2\kappa p_0}{n+1}r^2\right) dr, \tag{160}$$

where

$$h_1 = \int \frac{\nu' dr}{A + \frac{r\nu'}{2}}, \tag{161}$$

$$\alpha = \frac{1-3n}{5n+1}, \tag{162}$$

$$A = \frac{5n+1}{n+1}, \tag{163}$$

$$\beta = \frac{4n\,(9n+1)}{(n+1)\,(5n+1)}. \tag{164}$$

We shall study mainly the range $0 \leq n \leq \infty$ and occasionally some negative n. The case $n = -1/5$ demands special treatment to be done at the end of the section. The case $n = -1$ was discussed in Sec. IV. Each of the coefficients α, β and A has the same limits when $n \to \pm\infty$. When $0 \leq n \leq \infty$ their ranges are $-3/5 \leq \alpha \leq 1$, $1 \leq A \leq 5$, $0 \leq \beta \leq 36/5$. Expressions (159), (160) are generic for the cases $(\nu, q), (\nu, n), (\nu, \rho)$. The case (ν, q), discussed in the previous section, is obtained by putting $\alpha = -3$, $\beta = 2$, $A = 1$, changing the sign of H and an obvious replacement for q instead of p_0. The case (ν, ρ) will be studied in the next section.

In the present case Eq. (162) shows that $\alpha + 1$ is always positive for $n \in [0, \infty]$. Therefore, z will have a pole unless $C = 0$. Hence

$$z = e^F H. \tag{165}$$

After z is found, Eqs. (9),(10),(17) allow to extract the fluid components from the metric

$$\kappa\rho = \frac{1}{n+1}\left[\kappa p_0 + \frac{z}{r}\left(\nu' - \frac{z'}{z}\right)\right], \tag{166}$$

$$\kappa p = \frac{1}{n+1}\left[\frac{nz}{r}\left(\nu' - \frac{z'}{z}\right) - \kappa p_0\right], \tag{167}$$

$$\frac{q^2}{r^2} = 1 - z - \frac{r^2}{n+1}\left[\kappa p_0 + \frac{z}{r}\left(\nu' + n\frac{z'}{z}\right)\right]. \tag{168}$$

Since $p_0 > 0$, a necessary condition for positive pressure is $\nu' > z'/z$ and this leads to $\rho > \frac{p_0}{n+1}$. It is more convenient to express p and q from Eqs. (17), (84) respectively and lay all the difficulty in the calculations on the density. Plugging Eqs. (159), (160), (165) into Eq. (166) we obtain

$$(n+1)\,\kappa\rho = \kappa p_0 + \frac{4\kappa p_0}{(n+1)\left(A+\frac{r\nu'}{2}\right)} + \frac{2}{r^2}\left[\frac{z}{A} - \frac{1}{A+\frac{r\nu'}{2}}\right] + \left(2 - \frac{\beta}{A+\frac{r\nu'}{2}}\right)\frac{z\nu'}{r} + \frac{z\,(r\nu')'}{r\left(A+\frac{r\nu'}{2}\right)}, \tag{169}$$

which is an analog of Eq. (86) and leads to similar conclusions for ν. The poles in the two terms in the square brackets cancel when $r \to 0$ because $z \to 1$ and $\nu' \to 2\nu_0 r$. This condition makes ρ a well-defined function. We shall use for ν an ansatz which generalizes those in Eqs. (115), (145)

$$e^{\nu} = (a + br^s)^k\,, \tag{170}$$

where $s \geq 2$ is not necessarily an integer. We obtain for z

$$z = \frac{4\int (a+bx)^{k-1}\left[2aA + (2A+ks)\,bx\right]^{\gamma}\left(1 - \frac{2\kappa p_0}{n+1}x^{2/s}\right)x^{\frac{\alpha+1}{s}-1}dx}{sx^{\frac{\alpha+1}{s}}\,(a+bx)^{k-2}\left[2aA + (2A+ks)\,bx\right]^{1+\gamma}}, \tag{171}$$

$$\gamma = 1 - \frac{2k\beta}{2A+ks}. \tag{172}$$

There are several ways to simplify the expression under the integral. One of them is to set γ (which is not an integer, in general) equal to zero for any n choosing k and s appropriately. From Eq. (172) the following relation arises

$$\frac{s}{2} = \frac{(36k-25)\,n^2 + (4k-10)\,n - 1}{k\,(n+1)\,(5n+1)}. \tag{173}$$

The R.H.S. should be ≥ 1 which yields the condition $n \geq n_k$

$$n_k = \frac{k + 5 + 4\sqrt{k\,(2k+1)}}{31k - 25}. \tag{174}$$

The first few values are $n_1 \approx 2.15$, $n_2 \approx 0.53$, $n_3 \approx 0.39$, $n_4 = 1/3$. The first is beyond the physical range, the fourth is an exact number. One cannot probe the region $n < n_\infty \approx 0.21$ with this method.

Let us replace now z from Eq. (171) (with $\gamma = 0$) into Eq. (169) and calculate $\rho\,(0)$. When $s > 2$ the last two terms do not contribute, while from z only the term proportional to $a^{k-1}p_0$ counts. The result is very simple

$$\rho\,(0) = \frac{3p_0}{3n+1}. \tag{175}$$

The equation of state at the centre reads $\rho\,(0) + 3p\,(0) = 0$ which is exactly the equation of state for the ESU [50,52]. The pressure is negative at the centre, which is unrealistic.

In this connection it is worth to point out another relation to ESU even when γ does not vanish. Let us take in ν and z the limit $b = 0$. Then

$$e^{\nu} = a^k, \tag{176}$$

$$z = 1 - \frac{\kappa p_0}{3n+1}r^2, \tag{177}$$

ρ is given by Eq. (175) and $q = 0$. Now we obtain ESU for any r independently of s, k, n and p_0. Hence, the results in this section may be considered as a specific 'electrification' of ESU like the generalization of de Sitter solution in Sec.IV and of flat spacetime in Sec.V.

Going back to the case $\gamma = 0$, we reach the conclusion that $s = 2$ is a necessary condition for realistic solutions. The overlapping bands of solutions with $n \geq n_k$ collapse to an infinite set of discrete values n_k, $k \geq 2$. Then the ansatz

(170) coincides with (145) and we shall use the corresponding notation. A series of models is obtained, parameterized by k and n is given by Eq.(174). They have

$$e^{\nu} = a^k (1+t)^k , \tag{178}$$

$$z = \frac{\int (1+t)^{k-1} t^{\frac{\alpha+1}{2}-1} \left(1 - \frac{2}{n+1}\mu t\right) dt}{t^{\frac{\alpha+1}{2}} (1+t)^{k-2} \left[A + (A+k) t\right]}, \tag{179}$$

where $\mu = \kappa p_0/\tau$. Eq. (169) becomes

$$(n+1) \frac{\kappa\rho}{\tau} = \mu + \frac{4\mu (1+t)}{(n+1) \left[A + (A+k) t\right]} + \frac{2}{t} \left[\frac{z}{A} - \frac{1+t}{A+(A+k) t}\right] + \frac{2kz}{1+t} \left[2 - \beta \frac{1+t}{A+(A+k) t}\right] + \frac{4kz}{(1+t) \left[A+(A+k) t\right]}. \tag{180}$$

Now additional terms contribute to $\rho(0)$ and $p(0)$ and they can be made positive by a proper choice of τ and p_0. The pressure is given by Eq. (17) and the charge function by Eq. (84) which becomes

$$\tau q^2 = t \left[1 - \frac{1+(2k+1) t}{1+t} z + \frac{\kappa p}{\tau} t\right]. \tag{181}$$

Let us discuss the particular case $k = 2$, $n_2 = 0.53$. Then

$$z = \frac{1.19 + 0.35t - 0.46\mu t - 0.27\mu t^2}{1.19 + 2.19t} \tag{182}$$

and it can be shown that $p(0) > 0$ when $\mu < 6.34$. Let us take $\mu = 1$. Then

$$\frac{\kappa p}{\tau} = \frac{-4.05 - 6.1t + \frac{4z}{1+t} (4.57 + 6.57t)}{6.87 + 12.64t}, \tag{183}$$

$$\tau q^2 = t \left(1 - \frac{1+5t}{1+t} z + \frac{\kappa p}{\tau} t\right). \tag{184}$$

The pressure is monotonously decreasing and vanishes at $t_0 = 0.66$. The charge function reaches a maximum before t_0 and is still positive at t_0. This indicates that σ from positive turns to negative. As in Eq. (153) the condition $z(r_0) = e^{\nu}(r_0)$ determines a. The charge and the mass are determined by Eqs. (45),(46). All constants may be written as functions of r_0: $a = 0.37$, $b = 0.24/r_0^2$, $m = 0.32r_0$, $e = 0.49r_0$. The ratio $e/m > 1$.

Another interesting case is $k = 4$ with $n_4 = 1/3$ exactly. When $p_0 = 0$ and $q = 0$ this is the equation of state of pure incoherent radiation and the case is not integrable [46]. This specific charged version, which is not the only possible one, however, is integrable and assuming $\mu = 2/3$ we have

$$z = \frac{1 + \frac{2}{3}t - \frac{2}{7}t^3 - \frac{1}{9}t^4}{(1+t)^2 (1+3t)}, \tag{185}$$

$$\frac{\kappa p}{\tau} = \frac{5+9t}{12(1+3t)} + \frac{3(3+7t) z}{2(1+t)(1+3t)} - \frac{\frac{7}{3} + 3t + \frac{9}{7}t^2 + \frac{1}{9}t^3}{4(1+t)^2 (1+3t)} - \frac{2}{3}, \tag{186}$$

$$\tau q^2 = t \left(1 - \frac{1+9t}{1+t} z + \frac{\kappa p}{\tau} t\right). \tag{187}$$

The pressure is positive, monotonously decreasing and vanishes at $t_0 = 0.54$. The charge density has a negative part, like the previous case. The junction conditions yield $a = 0.3$, $b = 0.16/r_0^2$, $m = 0.5r_0$, $e = 0.45r_0$. This time $e/m < 1$. Probably, all models in the series have a range of μ where they are physically realistic.

The method used to simplify Eq. (160), after all, resembles the Korkina-Durgapal method, which was extended to the charged case (ν, q) in the previous section. With $s = 2$ the ansatz (170) is equivalent to (145) and, effectively, the non-integer γ was exchanged for an integer k in Eq. (171), which allows integration in rational functions. There are other ways to simplify Eq. (171), even when $k = 1$. One of them is to take again $n = 1/3$. Then $\alpha = 0$, $A = 2$, $\beta = 3/2$, $\gamma = 1/2$. We have

$$e^{\nu} = a\,(1+t)\,, \tag{188}$$

$$z = \frac{(1+t)\int (2+3t)^{1/2}\left(1-\frac{3}{2}\mu t\right)t^{-1/2}dt}{t^{1/2}\,(2+3t)^{3/2}}. \tag{189}$$

The integral is expressed in elementary functions

$$z = \frac{1+t}{t^{1/2}\,(2+3t)^{3/2}}\left[-\frac{1}{4}\,(\mu - 4 + 3\mu t)\sqrt{(2+3t)\,t} + \frac{4+\mu}{12}\sqrt{3}\ln\left(3t+1+\sqrt{3t\,(2+3t)}\right)\right], \tag{190}$$

where the integration constant was chosen so that $z\,(0) = 1$. The pressure and the charge are given by Eq. (17), (169) and (181)

$$\frac{\kappa p}{\tau} = -\frac{3\,(1+2t)}{4\,(2+3t)}\mu + \frac{9z}{4\,(2+3t)} + \frac{1}{2t}\left(\frac{z}{2} - \frac{1+t}{2+3t}\right), \tag{191}$$

$$\tau q^2 = t\left(1 - \frac{1+3t}{1+t}z\right) + \frac{\kappa p}{\tau}t^2. \tag{192}$$

The pressure is well-behaved for any μ and vanishes at t_0. However, q^2 also has a zero, but at t_1, and becomes negative in a region where $t > t_1$. Unfortunately, for most values of μ the inequality $t_1 < t_0$ holds, meaning that $q^2 < 0$ in some part of the interior. This is grossly unrealistic. We have been unable to find by computer simulations any realistic solution.

Another case with $k = 1$ which leads to simple functions can be guessed from Eq. (164). When $n = -1/9$, $\beta \equiv 0$. We have also $\alpha = 3$, $A = 1/2$, $\gamma = 1$. Although the pressure is negative, it is worth being discussed. Eqs. (188), (192) still hold. The integral in Eq. (171) is elementary and gives

$$z = \frac{1+t}{(1+3t)^2}\left[1 + \left(2 - \frac{3}{2}\mu\right)t - \frac{27}{8}\mu t^2\right]. \tag{193}$$

Eq. (169) leads to ρ and p similar in form, since Eq. (17) is satisfied. Like usual, we give only the formula for the pressure

$$\frac{\kappa p}{\tau} = \frac{1+t}{2\,(1+3t)^2}\left[2 + \frac{3}{2}\mu + \left(3 + \frac{27}{8}\mu\right)t\right] - \frac{9\,(1+2t)}{4\,(1+3t)}\mu - \frac{3z}{2\,(1+3t)}. \tag{194}$$

The values at the centre are

$$\frac{\kappa p}{\tau}\,(0) = -\frac{1}{2}\,(1+3\mu)\,M,\ \frac{\kappa \rho}{\tau}\,(0) = \frac{9}{2}\,(1+\mu)\,. \tag{195}$$

On the other hand

$$\frac{\kappa p}{\tau}\,(\infty) = \frac{1}{6} - \frac{9}{8}\mu. \tag{196}$$

Thus, when $0 < \mu < 4/27$ the pressure is monotonously increasing, crosses the t axis and forms a boundary of fluid with tension inside. Let us take the γ-law with $\mu = 0$. The charge function q^2 is positive and increasing. The junction conditions give $a = 0.55$, $b = 0.11/r_0^2$, $m = 0.23 r_0$, $e = 0.35 r_0$ where $t_0 = 0.2$. This solution resembles the classical models of the electron in Sec.IV.

Finally, some words should be said about the case $n = -1/5$, which is really very special. Eqs. (159), (160) are replaced by

$$e^F = \frac{4e^{-2h_2}}{r^6 e^\nu \left(r\nu'\right)^2}, \tag{197}$$

$$H = \frac{1}{2}\int e^\nu \nu' r^6 \left(2 - 5\kappa p_0 r^2\right) e^{2h_2} dr, \tag{198}$$

$$h_2 = 2\int \frac{dr}{r^2 \nu'}. \tag{199}$$

Let us choose the simple ansatz (115) or (188) for ν, written as

$$e^\nu = a + bx = a\left(1 + \tau x\right). \tag{200}$$

Then Eqs. (197), (198) become

$$e^F = \frac{a+bx}{b^2 x^6} e^{1/\tau x}, \tag{201}$$

$$H = -\frac{b}{2}\int \left(\frac{2}{X^6} - \frac{5\kappa p_0}{X^7}\right) e^{-X/\tau} dX, \tag{202}$$

where $X = 1/x$. The integrals can be expressed through the $\mathrm{Ei}\left(X\right)$ function. For example

$$\int X^{-6} e^{-X/\tau} dX = e^{-1/\tau x} \sum_{k=1}^{5} \frac{\left(-\tau\right)^{-k+1} x^{6-k}}{(6-1)\ldots(6-k)} + \frac{\left(-\tau\right)^{-5}}{5!} \mathrm{Ei}\left(-\frac{1}{\tau x}\right). \tag{203}$$

Then z, which is given again by Eq. (165), will have poles for $x = 0$, not compensated by Ei. Thus, already z is ill-defined, although the case is formally soluble. In addition $n < 0$. When $p_0 = 0$ and $q = 0$, this case is soluble [46]. It is connected by a Buchdahl transformation to the de Sitter solution with $n = -1$ [56,57] and is best expressed in isotropic coordinates. This transformation, however, does not work for electrostatic fields and probably is not applicable to the present case of perfect fluid plus an electrostatic field. Nevertheless, some memories from the uncharged case seem to have been preserved in a rather twisted form in the presence of charge.

The only paper which comes near to the results of the present section seems to be Ref. [36] which belongs to the (ν, q) case and is mentioned in Sec.VI (see Eq. (156)). When m/r_0 is small, this solution approximately satisfies Eq. (17) with $n = 3/10$, value rather close to $1/3$. The connection to our solutions is not clear.

A final remark is in order. It may be tempting to take the limit $e \to 0$ in one of the solutions obtained here and try to find analytic expressions for the non-integrable at present cases of γ-law uncharged solutions [46]. However, for the ansatz (145) used in this section, this leads to $r_0 \to 0$ and to flat spacetime. These solutions do not allow point-like models and this seems to be property only of the CD solutions, discussed in the introduction. In the case of general ν, it seems that the above limit will produce charged solutions with zero total charge. This is suggested by the behavior of σ in some of the models, namely, its change of sign in the interior. The proper non-charged limit seems to be $q\left(r\right) \equiv 0$. Then, however, Eq. (9) relates ρ to λ and we obtain the field equations of the uncharged fluid. Hence, nothing is gained in comparison with the three approaches described in Ref. [46]. They lead to Abel differential equations of the second kind with few integrable cases.

VIII. THE CASE (ν, ρ)

This case arises as a special limit of (ν, n) when $n \to \infty$, $p_0 \to \infty$. One sees from Eq. (17) that in this limit $\rho \to \rho_0 = p_0/n$ which can be made a finite constant. More generally, the third main equation (15) is obtained from Eq. (19) plus the change $\frac{p_0}{n+1} \to \rho\left(r\right)$. Eqs. (162)-(164) give $\alpha = -3/5$, $A = 5$, $\beta = 36/5$. The constant C in Eq. (22) is zero again to prevent a pole at the centre. Eqs. (158)-(161), (165) yield

$$f_1 = -2g' + \frac{16g}{r} - 72y, \tag{204}$$

$$z=\frac{2e^{\frac{36}{5}h_3}}{r^{2/5}e^{\nu}\left(5+\frac{r\nu'}{2}\right)^2}\int e^{\nu}\left(5+\frac{r\nu'}{2}\right)r^{-3/5}e^{-\frac{36}{5}h_3}\left(1-2\kappa\rho r^2\right)dr, \tag{205}$$

$$h_3=\int\frac{\nu' dr}{5+\frac{r\nu'}{2}}. \tag{206}$$

The parameters n and p_0 have been exchanged effectively for the density function and in this respect Eq. (205) resembles Eqs. (142),(143) from the (ν,q) case. When ρ is constant we obtain a generalization of the Schwarzschild interior solution [58]. It is much more complicated than the other generalization with constant T_0^0, which imposes just the simple ansatz (50) on the metric and was discussed in Sec.III. As seen from Eq. (205), the case $\rho=0$ is non-trivial and leads to purely electromagnetic mass models, because the mass function and T_0^0 receive contribution only from the charge. Formally, it is an analog of the γ-law equation of state ($p_0=0$) from the previous section.

Let us use again the ansatz (170) for ν. Now Eq. (172) reads

$$\gamma=1-\frac{72k}{50+5ks}, \tag{207}$$

which is nullified when

$$s=\frac{72k-50}{5k}. \tag{208}$$

This expression always gives $s\geq 2$ for $k\geq 1$. The analog of Eq. (171) with $\gamma=0$ is

$$z=\frac{4\int\left(a+br^s\right)^{k-1}\left(1-2\kappa\rho r^2\right)r^{-3/5}dr}{r^{2/5}\left(a+br^s\right)^{k-2}\left[10a+\left(10+ks\right)br^s\right]}. \tag{209}$$

Now ρ is given and p is found from Eq. (10)

$$\kappa p=\frac{z}{r}\left(\nu'-\frac{z'}{z}\right)-\kappa\rho. \tag{210}$$

Let us calculate $p\left(0\right)$. When $s>2$, just two terms contribute and the result is simple

$$p\left(0\right)=-\frac{1}{5}\rho\left(0\right). \tag{211}$$

This is similar to the relation following from Eq. (175). The pressure and the density cannot be both positive at the centre, hence, we must take $s=2$. Then Eq. (208) shows that k is not an integer and we gain nothing, trying to simplify z. The method which lead to an infinite series of realistic solutions in the (ν,n) case seems rather useless here. Therefore, let us put $k=1$ in Eq. (171). Then $\gamma=-1/5$ and it becomes

$$z=\frac{1+t}{t^{1/5}\left(5+6t\right)^{4/5}}\int\frac{\left(1-\frac{2\kappa\rho}{\tau}t\right)dt}{t^{4/5}\left(5+6t\right)^{1/5}}. \tag{212}$$

When $\rho=\rho_0$ the integrals bring forth the hypergeometric function

$$z=\frac{1+t}{\left(1+\frac{6}{5}t\right)^{4/5}}\left[F\left(\frac{1}{5},\frac{1}{5},\frac{6}{5};-\frac{6t}{5}\right)-\frac{\kappa\rho_0}{3\tau}tF\left(\frac{1}{5},\frac{6}{5},\frac{11}{5};-\frac{6t}{5}\right)\right]. \tag{213}$$

The charge is given by Eq. (192), while the pressure follows from Eq. (210) written as

$$\frac{\kappa p}{\tau}=\frac{2z}{1+t}-2z_t-\frac{\kappa\rho}{\tau}. \tag{214}$$

Searching for an elementary solution, one may consider some sophisticated density profile, leading to a simple integral in Eq. (212). For example

$$\frac{\kappa\rho}{\tau}=\frac{1}{2t}\left[1-\left(1+\frac{6}{5}t\right)^{1/5}\left(1-b_1t+b_2t^2\right)\right] \tag{215}$$

leads to

$$z=\frac{1+t}{\left(1+\frac{6}{5}t\right)^{4/5}}\left(1-\frac{b_1}{6}t+\frac{b_2}{11}t^2\right), \tag{216}$$

where b_1, b_2 are positive constants. We have $\rho(0)>0$ when $b_1>6/25$ and ρ is decreasing when $b_2>\frac{6b_1}{25}+\frac{72}{625}$. Taking $b_1=5$, $b_2=2$ a solution with realistic z, ρ and p is obtained, having a boundary at $t_0=0.52$. Unfortunately, $q^2<0$ in the interior, although for larger t it becomes positive. Perhaps, further fine tuning of the density may lead to physically acceptable solutions.

IX. THE CASES (ρ, Q), (λ, ρ) AND (λ, Q)

Having in mind the results in the previous section, it is no wonder that the case (ρ, q), where there is control over the evasive charge function, is much more popular in the literature. Eq. (9) easily relates this case to (λ, ρ) and (λ, q), and Eq. (24) or (25) may be used as a third main equation. Eqs. (7),(8) define z in terms of ρ and q

$$z=1-\frac{\kappa}{r}\int_0^r \rho r^2 dr-\frac{1}{r}\int_0^r \frac{q^2}{r^2}dr. \tag{217}$$

Eq. (9) shows that ρ is regular when $z-1\sim r^2$ for small r. We accept the ansatz

$$z=1-ar^2-br^4. \tag{218}$$

It follows from Eq. (217) when $q=Kr^l$ with $l=2,3$ and

$$\rho=\rho_0-\rho_1 r^2, \tag{219}$$

where ρ_0, ρ_1 are non-negative constants.. When $l=2$ or $l=3$ we have respectively

$$z=1-\frac{1}{3}\left(\kappa\rho_0+K^2\right)x+\frac{\kappa\rho_1}{5}x^2, \tag{220}$$

$$z=1-\frac{\kappa\rho_0}{3}x+\frac{1}{5}\left(\kappa\rho_1-K^2\right)x^2. \tag{221}$$

The total charge and mass are given by

$$e=Kx_0^{3/2}, \tag{222}$$

$$m=\frac{1}{2}\left[a+\left(b+K^2\right)x_0\right]x_0^{3/2}. \tag{223}$$

When $\rho_1\neq 0$ this is a model of a gaseous sphere, both density and pressure vanish at the boundary [25,26]. We have $\rho_0=\rho_1 x_0$. When $\rho_1=0$, $\rho_0\neq 0$ this is a generalization of the incompressible fluid sphere for $l=3$ [22] or $l=2$ [23]. Finally, when $\rho_1=\rho_0=0$ we come to a model with electromagnetic mass [6]. The case $b\neq 0$ appears to be simpler, due to a peculiarity in Eq. (24) and leads to elementary functions. Eq. (50) demonstrates that the other case, $b=0$, belongs to the family of solutions with constant T_0^0. Eq. (24) becomes in terms of x

$$\left(1-ax-bx^2\right)y_{xx}-\left(bx+\frac{a}{2}\right)y_x-\left(\frac{b}{4}+\frac{q^2}{2x^3}\right)y=0. \tag{224}$$

The coefficient before y is constant when $l=3$ (or $K=0$). Let us study this case first. Eq. (221) gives

$$d\equiv-\left(\frac{b}{4}+\frac{q^2}{2x^3}\right)=\frac{1}{20}\left(\kappa\rho_1-11K^2\right). \tag{225}$$

A change of variables brings Eq. (224) to (see Eq.2.1.164 from Ref. [49])

$$y_{\xi\xi} + dy = 0, \tag{226}$$

$$\xi = \int_0^x \frac{dx}{\sqrt{1 - ax - bx^2}}. \tag{227}$$

This integral has several different expressions according to the signs of b and $a^2 + 4b$, but many properties of the solutions depend only on the facts that $\xi\,(0) = 0$, ξ and ξ_x are positive. Eq. (226) is easily solved and the last function to be determined, the pressure, is given by Eq. (10), suitably rewritten as

$$\kappa p = 4\sqrt{z}\frac{y_\xi}{y} - 2z_x - \kappa\rho. \tag{228}$$

At the junction

$$y\,(x_0) = \sqrt{z\,(x_0)}. \tag{229}$$

Then from Eq. (228) follows that the pressure vanishes when

$$2y_\xi\,(x_0) = z_x\,(x_0) \tag{230}$$

is satisfied. The charge is given by Eq. (222) while the mass in Eq. (223) becomes

$$m = \frac{1}{2}\left[\frac{\kappa\rho_0}{3} + \left(6K^2 - \kappa\rho_1\right)\frac{x_0}{5}\right]x_0^{3/2}. \tag{231}$$

When $\rho_1 = 0$ the mass is positive. When $\rho_1 \neq 0$ this assertion still holds because then $\rho_1 x_0 = \rho_0$. The positivity of the mass does not impose any conditions on the constants ρ_0, x_0 and K which determine the solution.

Inserting z and ρ in Eq. (228) yields

$$\kappa p = 4\sqrt{z}\frac{y_\xi}{y} - \frac{1}{3}\kappa\rho_0 + \frac{1}{5}\left(3\kappa\rho_1 + 2K^2\right)x. \tag{232}$$

Then

$$\kappa p\,(0) = 4\frac{y_\xi}{y}\,(0) - \frac{1}{3}\kappa\rho_0 \tag{233}$$

should be positive. To be more concrete, let us introduce the three classes of solutions of Eq. (226) for $d = 0$, $d < 0$ and $d > 0$ respectively:

$$y = C_1 + C_2\xi, \tag{234}$$

$$y = C_1 e^{-\sqrt{-d}\xi} + C_2 e^{\sqrt{-d}\xi}, \tag{235}$$

$$y = C_1 \sin\sqrt{d}\xi + C_2 \cos\sqrt{d}\xi. \tag{236}$$

The condition $y > 0$ and Eq. (233) lead to $C_1 > 0$, $C_2 > 0$ when $d \geq 0$ and to $C_2 > 0$, $C_2 > C_1$ when $d < 0$. Therefore, e^ν increases with r until it meets z which decreases. We have $y\,(0) < 1$.

1) Case $d = 0$. Eq. (225) means $\kappa\rho_1 = 11K^2$, while Eq. (221) gives $b = -2K^2$. We also have

$$x_0 = \frac{\kappa\rho_0}{11K^2} \tag{237}$$

and the solution is determined by two constants, ρ_0 and K. Eqs. (229),(230) result in

$$C_2 = \frac{1}{66}\kappa\rho_0, \tag{238}$$

$$C_1 = -\frac{1}{66}\kappa\rho_0\xi_0 + \sqrt{z_0}, \tag{239}$$

$$z_0 = 1 - \frac{5}{3}v^2,\ v \equiv \frac{\kappa\rho_0}{11\,|K|} \tag{240}$$

and y is given by Eq. (234). C_1 is real when $v \leq \sqrt{3/5}$. The charge is given by Eq. (222) and the mass is

$$m = \frac{4}{33}\kappa\rho_0 x_0^{3/2}, \tag{241}$$

leading to the ratio $|e|/m = 3/4v \geq \sqrt{15}/4$. Due to the negative b Eq.(227) reads

$$\xi = \frac{1}{\sqrt{-b}}\ln\left|-\frac{a+2bx}{2\sqrt{-b}} + \sqrt{z}\right| - \frac{1}{\sqrt{-b}}\ln\left|1 - \frac{a}{2\sqrt{-b}}\right|. \tag{242}$$

In Ref. [26] another expression was used, true only when

$$a^2 + 4b = \frac{121}{9}K^2\left(v^2 - \frac{72}{121}\right) < 0. \tag{243}$$

This inequality, however, does not hold for v close to $\sqrt{3/5}$. Plugging Eq. (242) into Eq. (239) results in

$$C_1 = -\frac{v}{6\sqrt{2}}\left[\ln\left(\frac{v}{6\sqrt{2}} + \sqrt{z_0}\right) - \ln\left|1 - \frac{11v}{6\sqrt{2}}\right|\right] + \sqrt{z_0}. \tag{244}$$

The pressure reads

$$\kappa p = \frac{4C_2\sqrt{z}}{C_1 + C_2\xi} - \frac{1}{3}\kappa\rho_0 + 7K^2 x. \tag{245}$$

The condition $p(0) > 0$ is fulfilled when $C_1 < 2/11$. This holds when v varies near $\sqrt{3/5} = 0.77$. Thus, there are solutions with positive pressure.

2) Case $d < 0$. This means $\kappa\rho_1 < 11K^2$ and the limiting case $\rho_1 = 0$ is possible. We shall discuss first the general case. Eqs. (221),(222),(231),(235) hold. Conditions (229), (230) fix the constants

$$C_1 = \frac{1}{2}e^{\sqrt{-d}\xi_0}\left(\sqrt{z_0} - \frac{z_x(x_0)}{2\sqrt{-d}}\right), \tag{246}$$

$$C_2 = \frac{1}{2}e^{-\sqrt{-d}\xi_0}\left(\sqrt{z_0} + \frac{z_x(x_0)}{2\sqrt{-d}}\right). \tag{247}$$

The pressure is given by Eq. (232)

$$\kappa p = 4\sqrt{-dz}\frac{C_2 e^{\sqrt{-d}\xi} - C_1 e^{-\sqrt{-d}\xi}}{C_2 e^{\sqrt{-d}\xi} + C_1 e^{-\sqrt{-d}\xi}} - \frac{1}{3}\kappa\rho_0 + \frac{1}{5}\left(3\kappa\rho_1 + 2K^2\right)x. \tag{248}$$

Positivity at the centre demands

$$\frac{C_2 - C_1}{C_2 + C_1} > \frac{\kappa\rho_0}{12\sqrt{-d}}. \tag{249}$$

A necessary condition is $C_2 > C_1$, satisfied when

$$z_x(x_0) = -\frac{\kappa\rho_0}{3} - \frac{2}{5}\left(K^2 - \kappa\rho_1\right)x_0 \tag{250}$$

is positive, which leads to $\kappa\rho_1 > 6K^2$. Introducing the variable

$$\Lambda = \frac{\kappa\rho_1}{K^2}, \tag{251}$$

we obtain for its range $6 < \Lambda < 11$. Then $b < 0$ and ξ is given again by Eq. (242). Eqs. (246), (247) become

$$C_1 = \frac{1}{2}\left(\sqrt{z_0} + \frac{z_x(x_0)}{2\sqrt{-b}}\right)^{\sqrt{\frac{d}{b}}}\left(\sqrt{z_0} - \frac{z_x(x_0)}{2\sqrt{-d}}\right)\left|1 - \frac{a}{2\sqrt{-b}}\right|^{-\sqrt{\frac{d}{b}}}, \tag{252}$$

$$C_2 = \frac{1}{2}\left(\sqrt{z_0} + \frac{z_x(x_0)}{2\sqrt{-b}}\right)^{-\sqrt{\frac{d}{b}}}\left(\sqrt{z_0} + \frac{z_x(x_0)}{2\sqrt{-d}}\right)\left|1 - \frac{a}{2\sqrt{-b}}\right|^{\sqrt{\frac{d}{b}}}. \tag{253}$$

Obviously, $C_2 > 0$ and $C_2 + C_1 > 0$, as required. Inequality (249) is still very complicated. It becomes simpler if $d = b$ but this gives $\Lambda = 3$ which is out of range. Another way to simplify it is to seek solution with $C_1 = 0$. Eq. (252) allows to rewrite this condition as

$$K^2 x_0^2 = \frac{9(11-\Lambda)}{27 + 9\Lambda - \Lambda^2}, \tag{254}$$

while Eq. (249) becomes

$$1 > -\frac{(\kappa\rho_1)^2 x_0^2}{144d}. \tag{255}$$

Plugging here the previous formula transforms Eq. (255) into

$$(\Lambda + 2)(\Lambda - 6) < 0, \tag{256}$$

satisfied when $-2 < \Lambda < 6$. This interval is adjacent to the allowed interval for Λ, hence, this solution also possesses negative pressure. We have been unable to find a positive pressure solution.

Let us discuss the subcase $\rho_1 = 0$, i.e. $\rho = \rho_0$. Now Eq. (250) shows that $z_x(x_0) < 0$, therefore, $C_2 < C_1$ and either Eq. (249) is not satisfied or $y(0) < 0$, which is completely unrealistic. The pressure is negative in this subcase. Eq. (222) holds as it is, while Eq. (223) reads

$$m = \left(\frac{1}{6}\kappa\rho_0 + \frac{3}{5}K^2 x_0\right) x_0^{3/2}. \tag{257}$$

We have $b = K^2/5 > 0$ and consequently $a^2 + 4b > 0$. Then Eq. (227) reads

$$\xi = \frac{1}{\sqrt{b}} \arcsin\frac{a + 2bx}{\sqrt{a^2 + 4b}} - \frac{1}{\sqrt{b}} \arcsin\frac{a}{\sqrt{a^2 + 4b}}. \tag{258}$$

In Ref. [26] the expression

$$\xi = \frac{1}{\sqrt{-b}} arc\sinh\frac{a + 2bx}{\sqrt{-a^2 - 4b}}, \tag{259}$$

which holds only when $b < 0$ and $a^2 + 4b < 0$ was used in all cases and this is incorrect. Formula (258) allows to make connection with the results of Ref. [6], where in addition $\rho_0 = 0$, so that the density vanishes and the mass arises entirely from the electrostatic field energy. Eq. (250) still shows that the pressure is negative. Eqs. (222) and (257) give the ratio

$$\frac{|e|}{m} = \frac{5}{3|K| x_0}. \tag{260}$$

Now $a = 0$ and

$$z = 1 - \frac{1}{5}K^2 x^2. \tag{261}$$

It is positive at the boundary when $|K| x_0 < \sqrt{5}$ and this means $|e|/m > \sqrt{5}/3$. The second term in Eq. (258) vanishes and combining it with Eq. (235) yields

$$y = C_1 \exp\left(-\sqrt{\frac{11}{2}} \arcsin\sqrt{b}x\right) + C_2 \exp\left(\sqrt{\frac{11}{2}} \arcsin\sqrt{b}x\right). \tag{262}$$

The passage to arccos scales $C_{1,2}$ and interchanges their places, giving Eq. (4.6) from Ref. [6].

3) Case $d > 0$. Now we have $\kappa\rho_1 > 11K^2$ which permits the limit $K = 0$. The two cases do not differ essentially. Eq. (236) gives y. The constants read

$$C_1 = \sqrt{z_0}\sin\sqrt{d}\xi_0 + \frac{z_x(x_0)}{2\sqrt{d}}\cos\sqrt{d}\xi_0, \tag{263}$$

$$C_2 = \sqrt{z_0}\cos\sqrt{d}\xi_0 - \frac{z_x(x_0)}{2\sqrt{d}}\sin\sqrt{d}\xi_0. \tag{264}$$

The pressure is given by a formula, similar to Eq. (248). Positivity at the centre is ensured by

$$\frac{C_1}{C_2} > \frac{\kappa\rho_0}{12\sqrt{d}}. \tag{265}$$

The value of $z_x(x_0)$ is positive, while $b < 0$. Hence, ξ is given by Eq. (242), like in Ref. [25]. Condition (265) is even more complicated than Eq. (249) and simplifications do not seem possible. This is true even for the uncharged case, discussed in the mentioned reference. We shall remark only that the simple case $\kappa\rho_1 = \kappa\rho_0 = x_0 = 1$, $K = 0$ has positive pressure profile. For small enough K, the charged case must behave the same way. This ends the discussion of the case $l = 3$.

Let us study next the case $l = 2$ when z is given by Eq. (220). Eq. (224) is not soluble unless $b \equiv -\kappa\rho_1/5 = 0$ meaning that the density is constant. Then Eq. (224) becomes a particular case of the hypergeometric equation

$$\chi(\chi-1)y_{\chi\chi} + [(\eta_1+\eta_2+1)\chi - \eta_3]y_\chi + \eta_1\eta_2 y = 0, \tag{266}$$

where $\chi = ax$, $\eta_1 + \eta_2 + 1 = 1/2$, $\eta_3 = 0$, $\eta_1\eta_2 = K^2/2a$. These relations give

$$\eta_1 = -\frac{1}{4} \pm \frac{1}{4}\sqrt{1 - \frac{8K^2}{a}} \tag{267}$$

and $\eta_{1,2} \in [-1/2, 0)$. The reality of η_1 is ensured by

$$\kappa\rho_0 > 23K^2. \tag{268}$$

Thus, in the charged case it is not possible to put $\rho = \rho_0 = 0$. This case was studied by Wilson [23] who did not recognize the appearance of the hypergeometric function. He assumed that the pressure was positive and developed a series expansion for y. Now, since $\eta_3 = 0$, the two fundamental solutions of Eq. (266) are [59]

$$y_1 = \chi F\left(\eta_1 + 1, \frac{1}{2} - \eta_1, 2; \chi\right), \tag{269}$$

$$y_2 = F\left(\eta_1, -\frac{1}{2} - \eta_1, \frac{1}{2}, 1 - \chi\right). \tag{270}$$

This is because $1 + \eta_1 + \eta_2$, $\pm\eta_{1,2}$ are not integers. The first solution is unphysical since e^ν vanishes at $r = 0$. The second has a finite limit

$$y_2(0) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2} - \eta_1\right)\Gamma(1 + \eta_1)}, \tag{271}$$

so a linear combination between the two solutions is a candidate for a regular metric.

In Sec. VI we have shown that when e^ν is given by the simple ansatz (145) with $k = 1$, z is expressed through hypergeometric functions. Here we see that, in a rather symmetrical way, when z is given by the similar ansatz (50) (or (220) with $\rho_1 = 0$), y becomes a hypergeometric function. In this paper the emphasis is laid on solutions in elementary functions, so we shall not pursue this issue further, except in the case where $K = 0$. Then Eq. (266) becomes

$$(\chi - 1)y_{\chi\chi} + \frac{1}{2}y_\chi = 0 \tag{272}$$

and is easily solved

$$e^{\nu} = \left[2\left(1 - ar^2\right)^{1/2} + a_1\right]^2. \tag{273}$$

This is the expected Schwarzschild interior solution [58], contrary to the claims in Ref. [23].

A model for a superdense star with the ansatz

$$z = \frac{1 - a_2 r^2}{1 + a_1 a_2 r^2} \tag{274}$$

has been discussed in the uncharged case for $a_1 = 2$ [60,61], $a_1 = 7$ [62] and a set of discrete values for a_1 [43]. Recently, it was studied for arbitrary a_1 both in the uncharged [44] and the charged case [45]. The charge function is chosen to be

$$q = \frac{K a_2 r^3}{1 + a_1 a_2 r^2}, \tag{275}$$

which simplifies considerably the equation for y. Introducing the variable

$$\eta = \left(\frac{a_1}{a_1 + 1}\right)^{1/2} \left(1 - a_2 r^2\right)^{1/2}, \tag{276}$$

one obtains

$$\left(1 - \eta^2\right) y_{\eta\eta\eta} - \eta y_{\eta\eta} + d_1 y_\eta = 0, \tag{277}$$

$$d_1 = a_1 + 2 - \frac{2K^2}{a_1}, \tag{278}$$

where $d_1 > 0$ is required.

Formally, Eq. (277) is a subcase of Eq. (224) with $a = 0$, $b = 1$, $x \to r$, $y \to y_\eta$, so we can use the machinery developed there. Eq. (227) gives $\xi = \arcsin \eta$. We may change the variable to $\delta = \arccos \eta$ because $\xi + \delta = \pi/2$ and Eq. (226) is invariant under this change. Hence, y_η is given by Eq. (236) and y may be found by integration. Use of trigonometric equalities helps to find an expression with two terms

$$y = a_4 \left\{ \frac{\cos\left[\left(\sqrt{d_1} + 1\right)\delta + a_3\right]}{\sqrt{d_1} + 1} - \frac{\cos\left[\left(\sqrt{d_1} - 1\right)\delta + a_3\right]}{\sqrt{d_1} - 1} \right\}, \tag{279}$$

where the notation $C_1 = -2a_4 \sin a_3$, $C_2 = 2a_4 \cos a_3$ has been used. This formula was found in Refs. [44,45] by resorting first to Gegenbauer functions and then to Chebyshev polynomials. The peculiarities of Eq. (224) permit a straightforward derivation in elementary functions.

Finally, let us discuss a solution of Nduka [35], belonging to the class (λ, q). He chose a constant $z \neq 1$ and generalized the uncharged solution of Ref. [34] by taking $l = 1$, i.e. $q = Kr$. Then Eq. (24) turns into the Euler equation. Repeating the analysis referring to Ref. [37], after Eq. (157), we see that y is singular at $r = 0$. The same is true for the density, $\rho \sim 1/r^2$.

X. THE CASE (λ, N)

In this case the fluid satisfies the linear equation of state (17) and some ansatz for λ is given. The other metric component is found from Eq. (27) - a linear second-order equation for y.

Let us study first the case $z = c < 1$ which leads to singular pressure and density, but provides the opportunity to generalize the well-known KT solution [41,42,46,63,64]. Eq. (27) becomes the Euler equation when $p_0 = 0$ (γ-law)

$$r^2 y'' + k_0 r y' - k_1 y = 0, \tag{280}$$

$$k_0 = \frac{3 - n}{n + 1}, \qquad k_1 = \frac{1 - c}{c}, \tag{281}$$

whose solution is

$$e^{\nu} = C_1 r^{1-k_0+2k_2}, \tag{282}$$

$$k_2 = \frac{1}{2}\left[\left(1-k_0\right)^2 + 4k_1\right]^{1/2}. \tag{283}$$

It can be checked that $1 - k_0 + 2k_2 > 0$. We have included only the non-divergent at $r = 0$ solution. The density and the charge function are given by Eqs. (166), (168) respectively

$$\left(n+1\right)\kappa\rho = \frac{\left(1-k_0+2k_2\right)c}{r^2}, \tag{284}$$

$$\frac{q^2}{r^2} = \left(1-c\right)\left[1 - \frac{4n+\left(n+1\right)^2}{\left(n+1\right)^2}c\right]. \tag{285}$$

The density is positive and has a pole at the centre. Obviously

$$c \leq \frac{\left(n+1\right)^2}{4n+\left(n+1\right)^2} \tag{286}$$

should hold. This solution is a charged generalization of the KT solution which is recovered when the equality holds. Due to $p_0 = 0$ it does not have a boundary and is not asymptotically flat. Another generalization was offered in Ref. [40] but it alters many of its properties, including the equation of state. If $p_0 \neq 0$ Eq. (27) becomes

$$r^2 y'' + k_0 r y' + \left(-k_1 + k_3 r^2\right) y = 0, \tag{287}$$

$$k_3 = \frac{2\kappa p_0}{\left(n+1\right)c}. \tag{288}$$

Its solution is given by Bessel functions [49]

$$y = r^{\frac{1-k_0}{2}}\left[C_1 J_{k_2}\left(\sqrt{k_3}r\right) + C_2 Y_{k_2}\left(\sqrt{k_3}r\right)\right]. \tag{289}$$

Let us next discuss the ansatz (50), imposed by the condition of constant T_0^0, like in Sec. III. Introducing $x = ar^2$ turns Eq. (27) into

$$x\left(x-1\right)y_{xx} + \left[\frac{n+5}{2\left(n+1\right)}x - \frac{2}{n+1}\right]y_x + Dy = 0, \tag{290}$$

$$D = \frac{1}{2\left(n+1\right)}\left(3n+1-\frac{\kappa p_0}{a}\right). \tag{291}$$

This is once again the hypergeometric equation and one of its fundamental solutions is $y_1 = F\left(\eta_1, \eta_2, \eta_3; x\right)$ where

$$\eta_2 = \varepsilon - \eta_1, \tag{292}$$

$$\eta_3 = \frac{2}{n+1}, \tag{293}$$

$$\eta_1\eta_2 = D, \tag{294}$$

$$\varepsilon = \frac{3-n}{2\left(n+1\right)}. \tag{295}$$

The solution is determined by the parameter n and the ratio $\kappa p_0/a$ which enters the definition of D. Eq. (294) shows that we can take instead η_1 as a second independent parameter. Eq. (291) provides one upper limit for D: $D \leq \frac{3n+1}{2(n+1)}$. In the physical range $0 \leq n \leq 1$ it increases from 1/2 to 1. Eq. (294) is a quadratic equation for η_1 and it is real when $D \leq (3-n)^2/16(n+1)^2$. This second limit decreases when n increases, from 9/16 to 1/16. There is a value $n_0 = 0.026$ where both limits meet. Thus, the first inequality is stronger when $n \in [0, n_0]$, while the second dominates for $n \in [n_0, 1]$. The interval $[0, n_0]$ is also the range of n where η_1 is real when $p_0 = 0$. The case $n = 1$ is degenerate because $\eta_3 = 1$. This affects the second fundamental solution [59] (Vol.1). When $n < 1$ it is

$$y_2 = x^{\frac{n-1}{n+1}} F\left(\eta_1 + \frac{n-1}{n+1}, \frac{1}{2} - \eta_1, \frac{2n}{n+1}; x\right) \tag{296}$$

and has a pole at $r = 0$. Therefore, we shall discard it. When $n = 1$ and $\eta_1, \frac{1}{2} + \eta_1$ are not integers, we have

$$y_2 = F\left(\eta_1, \frac{1}{2} - \eta_1, \frac{1}{2}; 1 - x\right). \tag{297}$$

This function diverges at $x = 0$ due to the formula

$$F(\eta_1, \eta_2, \eta_3; 1) = \frac{\Gamma(\eta_3)\Gamma(\eta_3 - \eta_2 - \eta_1)}{\Gamma(\eta_3 - \eta_1)\Gamma(\eta_3 - \eta_2)}. \tag{298}$$

When $\eta_1 = -k$ or $-k + \frac{1}{2}$, k being a positive integer, y_2 is given by the so-called Kummer series and is reduced essentially to a polynomial. When $\eta_1 = k$ or $k + \frac{1}{2}$ the same is true, due to the symmetry of F with respect to its first two arguments. In the following we shall use mainly y_1 which is well-defined by the hypergeometric series and converges for $x \in [0, 1]$. The generic case may be divided into three subcases: 1) $D = 0$, 2) $D < 0$ and 3) $D > 0$. We shall search them in turn for elementary solutions.

1) $D = 0$. Then $\eta_1 = 0$, $\eta_2 = \varepsilon$, $\kappa p_0/a = 3n + 1$ and y is expressed through the incomplete β-function

$$y = C_1 + C_2 \int x^{-\frac{2}{n+1}} (1-x)^{-1/2} dx. \tag{299}$$

When $n = 1$ the integral is elementary and reads

$$\ln \frac{1 - (1-x)^{1/2}}{1 + (1-x)^{1/2}}. \tag{300}$$

The logarithmic singularity at the beginning is obvious. When $n < 1$ it becomes a pole so that $C_2 = 0$ is required. Then, however, $q = 0$ and the solution is uncharged. It is easy to find that $\kappa p = -a$ and $\rho + 3p = 0$. We obtain ESU [50,52] and Eq. (299) represents its ill-defined generalization to the charged case.

2) $D < 0$. Ambiguity is fixed by accepting that $\eta_1 < 0$, $\eta_2 > 0$. A rather special subcase is $\eta_1 = -1/2$. Then $\eta_2 = \eta_3$ and

$$e^{\nu} = 1 - x = z. \tag{301}$$

Eq. (10) gives $\rho + p = 0$ while $q = 0$, $\rho = \frac{p_0}{n+1}$. Thus for any n we obtain the de Sitter metric whose charged generalizations were discussed in Sec. IV. This effective change in the equation of state was necessary because Eq. (27) is undefined for $n = -1$, and is possible when ρ and p are constant. It was noticed in the uncharged case for ESU in Ref. [47].

Another interesting elementary subcase is given by $\eta_1 = -k$ when F becomes a polynomial obtained from the first $k + 1$ terms of the hypergeometric series. It may be written also as

$$F(-k, \varepsilon + k, \eta_3; x) = \frac{k!}{(\eta_3)_k} P_k^{(\eta_3 - 1, -1/2)}(1 - 2x) = \frac{(1/2)_k \, k!}{(\eta_3)_k (\varepsilon)_k} C_{2k}^{(\varepsilon)}\left(\sqrt{1-x}\right), \tag{302}$$

where $(a)_k = a(a+1)...(a+k-1)$, P and C are respectively Jacobi and Gegenbauer orthogonal polynomials [59] (Vol.2). Unfortunately, this nice solution is plagued, like the de Sitter solution, and like all solutions with negative D, by tension. To show this let us plug our ansatz in Eqs. (166)-(168) and replace $\kappa p_0/a$ by D through Eq. (291):

$$\frac{(n+1)q^2}{ar^4} = 2(n+1)D - 2(1-x)\nu_x, \tag{303}$$

$$\frac{(n+1)\,\kappa p}{a}=-\,(n+1)+2\,(n+1)\,D+2n\,(1-x)\,\nu_x. \tag{304}$$

Since $D<0$, $q^2>0$ only if $\nu_x<0$. But then all three terms in the pressure are negative.

3) $D>0$. Then only the combination $\eta_{1,2}>0$ is possible, for when $\eta_1<0$, $\eta_2>0$. Eqs. (292), (294) yield $0<\eta_1<\varepsilon$. The right limit decreases with n and always $\varepsilon\leq 3/2$. An interesting subcase is $n=1/3$. We have $\varepsilon=1$, $\eta_3=1/2$, $D=\eta_1\,(1-\eta_1)$. Then y_1 degenerates into

$$y=C_1\frac{\sin\left(1-2\eta_1\right)R}{\left(1-\eta_1\right)\sin 2R}, \tag{305}$$

$$x=\sin^2 R. \tag{306}$$

The ranges are $0<\eta_1<1$, $0<D<1/4$, $4/3<\kappa p_0/a<2$. When η_1 becomes bigger than $1/2$ the sign of C_1 should be changed to keep y positive. Eqs. (303), (304) become

$$\frac{q^2}{3ar^4}=\frac{2}{3}D-\frac{1}{2}\,(1-x)\,\nu_x, \tag{307}$$

$$\frac{\kappa p}{a}=\frac{1}{2}\,(1-x)\,\nu_x-1+2D. \tag{308}$$

We demand the positivity of p. This means at least

$$\frac{1}{2}\,(1-x)\,\nu_x>\frac{1}{2}, \tag{309}$$

i.e. ν_x must be positive. Then, however, the first term in Eq. (307) cannot compensate the second because $2D/3<1/6$. If we make q^2 positive, the pressure becomes negative in complete analogy with the case $D<0$.

Finally, let us point out that Eq. (27) may be obtained by linearization of Eq. (33). Let us search for simple solutions of this equation. In Sec. IV we solved the case $n=-1$, when $Y=0$ using Eq. (30). Let us assume now that

$$\frac{2}{r^4}\left(r^2M'\right)'=\frac{4\kappa p_0}{(n+1)\,r}. \tag{310}$$

Quite interestingly, this again leads to the ansatz (50)

$$z=1-\frac{\kappa p_0}{3\,(n+1)}r^2 \tag{311}$$

and $\kappa p_0/a=3\,(n+1)$, $D=-\frac{1}{n+1}$. Then Eqs. (292), (294) give

$$\eta_1=\frac{3-n\pm(n+5)}{4\,(n+1)}, \tag{312}$$

so that η_1 and n are not independent. The first root leads to $\eta_1=\eta_3$ and Eq. (301). This is the trivial $Y=0$ solution of Eq. (33). The other root gives $\eta_1<0$ and $D<0$, leading to negative pressure. Something more, from the general solution of Eq. (33)

$$Y=\frac{1}{z^{1/2}r^{\frac{4}{n+1}}\left(C+\frac{1}{2}\int z^{-3/2}r^{\frac{n-3}{n+1}}dr\right)}, \tag{313}$$

it is seen that Y (and the pressure) has a pole at the centre, even when $C=0$. In conclusion, we have been unable to find realistic solutions with positive pressure for the ansatz (50).

XI. DISCUSSION AND CONCLUSIONS

Charged static perfect fluids have attracted considerably less attention than the uncharged ones, the number of papers being roughly an order of magnitude smaller. Solutions have been rare and the introduction hints how diverse were the approaches to their study. Charged dust occupies the first place in popularity and puts forth the idea of point-like particles. The rest is a mixture of generalizations: of the Schwarzschild idea about incompressibility, which may mean constant T_0^0 or constant density in the charged case, with the limiting case of vanishing density and electromagnetic mass models, generalization of the idea about vacuum polarization which brings forth the de Sitter solution, generalization simply of well-known uncharged solutions like those of Adler, Kuchowicz, Klein, Mehra, Vaidya-Tikekar and others, generalization of Weyl type connections, well studied in the electrovac case.

In this paper we have tried to show that the charged case has a life of its own when subjected to a natural and organized classification scheme. Surprisingly, in many respects it looks simpler than the uncharged case. The presence of the charge function serves as a safety valve, which absorbs much of the fine tuning, necessary in the uncharged case. The general formulae derived here show that the abundance of elementary solutions is probably bigger than in the traditional case. The proposed scheme becomes rather trivial there, representing ansatze mainly for λ, ν and sometimes ρ. In the charged case, however, it allows to sort out the different ideas mentioned above. Thus constant T_0^0 leads to the cases $\left(\lambda = 1 - ar^2, *\right)$, while constant density - to the much more difficult cases (ν, ρ) and (λ, ρ), similar to (ν, n) and (λ, n). Electromagnetic mass models are subcases, often spoiled by negative pressure. The point-like idea seems to be viable only for CD, leading to flat spacetime when the pressure does not vanish. The generalization of uncharged solutions firmly occupies the cases (λ, q) and (ν, q). Models with $n = -1$ are intimately related to the soluble case $n \neq -1$ and are much richer than their traditional protagonist - the de Sitter solution. The other models with negative pressure, which seem worth being studied, are the generalizations of ESU with $n = -1/3$. Finally, the Weyl type relations, so successful in electrovac and CD environments, seem rather out of place in a 'pressurized' perfect fluid.

Another advantage of this classification is that it delineates the degree of difficulty and the most convenient points for attack of the problem. The easiest cases seem to be (λ, Y) and (ν, Y). If we insist on authentic fluid characteristics and not their combinations, then (ν, p) and (ν, q) are the best candidates. On the other hand, the most difficult are (ρ, p) and (p, q), which seem to be accessible only numerically. The most unpredictable case is probably (λ, ν) since there is no control on exactly those characteristics which must satisfy the majority of regularity and positivity conditions.

A different division is between general and special cases. As was noticed in the introduction, a known solution belongs to any of the general cases; a solution of (ν, q) may be written as (λ, p) or even as (ρ, p) solution. Only the simplicity of two of the five functions ν, λ, ρ, p, q betrays where was the starting point. The special cases we discussed, apart from the ansatz (50), are essentially three: $\rho + p = 0$ and (λ, n), (ν, n). All of them have a linear equation of state. It should be mentioned that all integrable uncharged solutions with γ-law found in Ref. [46] have their charged generalizations with $p_0 \neq 0$ and, hence, have a boundary. The case $n = 0$, $p_0 = 0$ represents CD and arises as an electrification of the trivial uncharged dust solution (flat spacetime) as seen from Eqs. (95)-(98). Its boundary may be put anywhere, since $p = 0$ everywhere. The case $n = -1$ generalizes the de Sitter solution into a bunch of new solutions, characterized by the mass function, subjected to several mild restrictions. It was completely solved in Sec. IV. A second, this time ill-defined generalization, is given by Eq. (313). The de Sitter solution appears also as a special case in Sec. X (the class $D < 0$ with $\eta_1 = -1/2$). Since Eq. (33) excludes the case $n = -1$ this becomes possible thanks to an effective change in the equation of state. The option $n = -1/3$ in the uncharged case is a mark of ESU. One possible generalization of it was performed in Sec. VII (see Eqs. (176), (177) and their discussion). The ESU appears also as the special class $D = 0$ in Sec. X. The case $n = -1/5$ is special in the charged theory too, but its metric is singular, as was found in Eqs. (202), (203). The KT solution is generalized in Eqs. (282)-(286). In the uncharged case these exhaust the integrable cases of a complicated Abel differential equation of the second kind. In the charged case it is replaced by a linear first-order Eq. (19), which has many other solutions, discussed in Sec. VII, or by a linear second-order Eq. (27), discussed in Sec. X.

From a mathematical viewpoint, the charged case delivers a surprising variety of equations like those of Euler (157), (280), Bernoulli (75), (313), Riccati (30)-(33), Emden-Fowler (113), Abel (37), (102), (114) and the hypergeometric equation (266), (290). The cases containing λ quickly lead to special functions whose spectrum is also rich: Ei, *cn*, Bessel, hypergeometric, incomplete β-functions, Jacobi, Gegenbauer and Chebyshev polynomials, etc. Therefore, a more thorough study would require computer simulations of special functions, which are easier than purely numeric simulations.

From a physical viewpoint, the most interesting results are, in our opinion, the following:

1) The case of constant T_0^0 is solved by a simple algorithm involving algebraic operations, one differentiation and one simple integration upon a generating function Y, which satisfies few simple inequalities, Eqs. (51)-(53).

2) The charged de Sitter case is completely soluble in terms of the mass or the charge function, Eqs. (56)-(59). An example is given by Eqs. (71)-(74). There is a generalization with positive pressure in Eqs. (76)-(78). An example is given by Eqs. (79)-(82).

3) The general solution for a given pressure may be written as three contributions to z; from regular CD, from general CD and from the pressure, see Eq. (83). There is a general CD solution which creates a halo around charged fluid balls and may postpone their junction to a RN solution. An example is given by Eqs. (115)-(140).

4) The solutions of Korkina-Durgapal possess realistic charge generalizations given by Eqs. (142)-(145). An example is given by Eqs. (148)-(153).

5) Fluids with linear equation of state are integrable for any n as seen from Eqs. (19), (27), (33), (159)-(165). Elementary solutions may be found by several simplification techniques. Physically realistic is an infinite series of models given by Eqs. (173), (174), (178)-(180). Realistic examples are given by Eqs. (182), (185). The status of the model given by Eq. (190) is still unknown.

6) The case (ν,ρ) is closely connected to (ν,n) and is much more difficult than its companion with constant T_0^0, even when the density is constant or zero. An example with special functions is given by Eq. (213), another one - by Eq. (216). Their physical status is still undetermined.

7) The (ρ,q) case discussed in Ref. [26] is incomplete and has errors. It is treated in detail in Sec. IX and its connection with Ref. [6] is elucidated. It has several realistic cases, but electromagnetic mass models all seem to have negative pressure. Realistic examples are given by Eqs. (218), (234), (238)-(245) and Eqs. (263)-(265) with the values specified after them. A formal, but intriguing parallel is drawn to star models with simple spatial geometry [45].

8) The solution of Wilson [23] is expressed by a hypergeometric function and reduces to the Schwarzschild interior solution when the charge vanishes. Its realistic status is not known.

9) The KT solution has a charged generalization which is likewise singular, but may serve as a focal point to some regular solution, see Eqs. (282)-(286).

10) Solutions of the case (λ,n) with constant T_0^0 are expressed in hypergeometric functions, see Eqs. (290), (291). All degenerate elementary solutions have either negative pressure or singularities.

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[1] W. B. Bonnor, Zeit. Phys. **160**, 59 (1960).
[2] U. De and A. K. Raychaudhuri, Proc. Roy. Soc. A **303**, 97 (1968).
[3] A. K. Raychaudhuri, Ann. Inst. Henri Poincare A **22**, 229 (1975).
[4] P. S. Florides, Nuovo Cim. A **42**, 343 (1977).
[5] H. J. Efinger, Zeit. Phys. **188**, 31 (1965).
[6] F. I. Cooperstock and V. de la Cruz, Gen. Relat. Grav. **9**, 835 (1978).
[7] S. D. Majumdar, Phys. Rev. **72**, 390 (1947).
[8] A. Papapetrou, Proc. Roy. Irish Acad. A **51**, 191 (1947).
[9] A. Das, Proc. Roy. Soc. A **267**, 1 (1962).
[10] W. B. Bonnor, Mon. Not. Roy. Astr. Soc. **129**, 443 (1965).
[11] W. B. Bonnor and S. B. P. Wickramasuriya, Int. J. Theor. Phys. **5**, 371 (1972).
[12] W. B. Bonnor and S. B. P. Wickramasuriya, Mon. Not. Roy. Astr. Soc. **170**, 643 (1975).
[13] W. B. Bonnor, Class. Quantum Grav. **16**, 4125 (1999).
[14] W. B. Bonnor, Class. Quantum Grav. **15**, 351 (1998).
[15] A. Banerjee and M. M. Som, Int. J. Theor. Phys. **20**, 349 (1981).
[16] M. Gürses, Phys. Rev. D **58**, 044001 (1998).
[17] M. Gürses, e-print, gr-qc/9806038 (1998).
[18] V. Varela, e-print, gr-qc/9911062 (1999).
[19] R. Gautreau and R. B. Hoffman, Nuovo Cim. B **16**, 162 (1973).
[20] D. Ida, Prog. Theor. Phys. **103**, 573 (2000).
[21] B. S. Guilfoyle, Gen. Relat. Grav. **31**, 1645 (1999).
[22] C. F. Kyle and A. W. Martin, Nuovo Cim. **50**, 583 (1967).
[23] S. J. Wilson, Can. J. Phys. **47**, 2401 (1969).
[24] F. de Felice, L. Siming, Y. Yanqiang, Class. Quantum Grav. **16**, 2669 (1999).
[25] A. L. Mehra, J. Austr. Math. Soc. **6**, 153 (1966).
[26] A. L. Mehra, Gen. Relat. Grav. **12**, 187 (1980).
[27] J. M. Cohen and M. D. Cohen, Nuovo Cim. **60**, 241 (1969).
[28] R. N. Tiwari, J. R. Rao and R. R. Kanakamedala, Phys. Rev. D **30**, 489 (1984).

[29] Ø. Grøn, Phys. Rev. D **31**, 2129 (1985).
[30] R. Gautreau, Phys. Rev. D **31**, 1860 (1985).
[31] W. B. Bonnor and F. I. Cooperstock, Phys. Lett. A **139**, 442 (1989).
[32] F. I. Cooperstock and N. Rosen, Int. J. Theor. Phys. **28**, 423 (1989).
[33] K. D. Krori and J. Barua, J. Phys. A **8**, 508 (1975).
[34] B. Kuchowicz, Acta Phys. Pol. **33**, 541 (1968).
[35] A. Nduka, Acta Phys. Pol. B **8**, 75 (1977).
[36] A. Nduka, Gen. Relat. Grav. **7**, 493 (1976).
[37] T. Singh and R. B. S. Yadav, Acta Phys. Pol. B **9**, 475 (1978).
[38] M. Wyman, Phys. Rev. **75**, 1930 (1949).
[39] R. J. Adler, J. Math. Phys. **15**, 727 (1974).
[40] M. Cataldo and N. V. Mitskiévič, Class. Quantum Grav. **9**, 545 (1992).
[41] O. Klein, Ark. Mat. Astr. Fys. A **34**, N19 (1947).
[42] R. C. Tolman, Phys. Rev. **55**, 364 (1939).
[43] S. D. Maharaj and P.G. L. Leach, J. Math. Phys. **37**, 430 (1996).
[44] S. Mukherjee, B. C. Paul and N. K. Dadhich, Class. Quantum Grav. **14**, 3475 (1997).
[45] R. Sharma, S. Mukherjee and S. D. Maharaj, Gen. Relat. Grav. **33**, 999 (2001).
[46] B. V. Ivanov, e-print, gr-qc/0107060 (2001).
[47] B. V. Ivanov, e-print, gr-qc/0107032 (2001).
[48] J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55**, 374 (1939).
[49] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations* (Boca Raton, FL, Chemical Rubber Company, 1995).
[50] M. S. R. Delgaty and K. Lake, Comput. Phys. Commun. **115**, 395 (1998).
[51] P. Wils, Gen. Relat. Grav. **22**, 539 (1990).
[52] M. R. Finch and J. E. F. Skea, unpublished preprint, www.dft.if.uerj.br/users/Jim Skea/papers/pfrev.ps.
[53] H. Heintzmann, Zeit. Phys. **228**, 489 (1969).
[54] M. P. Korkina, Sov. Phys. J **24**, 468 (1981).
[55] M. C. Durgapal, J. Phys. A **15**, 2637 (1982).
[56] H. A. Buchdahl, Austr. J. Phys. **9**, 13 (1956).
[57] E. N. Glass and S. P. Goldman, J. Math. Phys. **19**, 856 (1978).
[58] K. Schwarzschild, Sitz. Deut. Akad. Wiss. Math.-Phys. Berlin **23**, 189 (1916).
[59] *Higher Transcendental Functions*, ed. by A. Erdelyi et al (Mc Graw-Hill, New York, 1953).
[60] P. C. Vaidya and R. Tikekar, J. Astrophys. Astron. **3**, 325 (1982).
[61] M. C. Durgapal and R. Banerjee, Phys. Rev. D **27**, 328 (1983).
[62] R. Tikekar, J. Math. Phys. **31**, 2454 (1990).
[63] O. Klein, Ark. Fys. **7**, 487 (1954).
[64] C. W. Misner and H. S. Zapolsky, Phys. Rev. Lett. **12**, 635 (1964).